iJournals International Journal of Software & Hardware Research in Engineering
ISSN No: 2347-4890 Volume 2 Issue 11, November-2014

DUALITY THEOREMS AND THEOREMS OF THE ALTERNATIVE

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ABSTRACT

Farkas' lemma is a key result in optimization theory. It is one of the theorems of the alternative. These theorems characterize the optimality conditions of several minimization problems. A theorem of alternative asserts that either the primal system has a solution or the dual system has a solution, but never both. The optimality conditions are also given by the duality theorem. This paper is a review of references [1], [2]. Here we show that, using simple logical arguments, a duality theorem is equivalent to a theorem of the alternative. These arguments do not need assumptions of linear space structure. With this we prove an alternative form of Farkas' lemma (AFFL) using Tucker's theorem and show that various theorems of alternative are simply special cases of this alternative form. This AFFL is equivalent to Farkas' lemma.

Keywords

Farkas' lemma, Theorems of alternative, Duality theorem, Tucker's theorem.

1. INTRODUCTION

To prove a duality theorem linking two constrained optimization problem, a theorem of alternative is applied. Sometimes this theorem of alternative is called transposition theorem. It has been observed that from such a duality result a theorem of the alternative is followed.

Here we show a simple, basic and logical principle that a duality theorem is equivalent to a theorem of alternative. For this no linear space structure is needed.

Let X and Y be arbitrary non-empty sets and let f and g be arbitrary extended real valued functions defined on X and Y respectively. For each $\alpha \in (-\infty, +\infty]$,

consider the statements

(I_α) There exists $x \in X$ such that $f(x) < \alpha$

(II_α) There exists $y \in Y$ such that $g(y) \geq \alpha$.

The following logical statement is an abstract theorem of the alternative involving the pairs (f, X) and (g, Y) :

for all $\alpha \in (-\infty, +\infty]$, exactly one of (I_α), (II_α) holds. (1)

Consider two abstract optimization problems, the primal problem as

$$\inf_{x \in X} f(x) \tag{2}$$

and the dual problem as

$$\sup_{y \in Y} g(y) \tag{3}$$

For these problems, the abstract duality theorem is the following logical statement.

$$\inf_{x \in X} f(x) = \max_{y \in Y} g(y) \tag{4}$$

This statement means $\inf_X f = \sup_Y g$ and $\sup_Y g = g(y)$ for some y i.e., $\sup_Y g$ is attained in *Y*.

Neither (1) nor (4) has any real content until the pairs (f, X) and (g, Y) are assigned specific interpretations, or structure, and also hypotheses are given under which (1) or (4) is true. The theory of dual optimization problems involves representing f and g in terms of some other function $L : X \times Y \to [-\infty, +\infty]$ such that

$$f(x) = \sup_{y \in Y} L(x,y) \quad \text{for all } x \in X \qquad (5)$$

and

$$g(y) = \inf_{x \in X} L(x,y) \text{ for all } y \in Y \qquad (6)$$

Then the primal is

$$\inf_{x \in X} f(x) = \inf_{x \in X} \sup_{y \in Y} L(x,y) \qquad (7)$$

and the dual is

$$\sup_{y \in Y} g(y) = \sup_{y \in Y} \inf_{x \in X} L(x,y) \qquad (8)$$

The abstract duality theorem is

$$\inf_{x \in X} \sup_{y \in Y} L(x,y) = \sup_{y \in Y} \inf_{x \in X} L(x,y) \qquad (9)$$

This is called the abstract minimax theorem.

Note : The symbol $(\sim)$ denotes negation.

Lemma 1 : The following two statements are equivalent.

(a₁) for all $\alpha \in (-\infty, +\infty)$,

(II_α) implies $\sim I_\alpha$

$$[\equiv (\mathrm{I}_\alpha) \Rightarrow \sim II_\alpha = \sim (II_\alpha) \vee \sim (I_\alpha) = (II_\alpha \wedge I_\alpha)$$

(d₁) [Weak duality theorem] $\inf_{x \in X} f(x) \geq \sup_{y \in Y} g(y)$

Proof : [(a1) ⇒ (d1)]

Let and put . If then obviously . Since is arbitrary and LHS is independent of y, . Now if , by (a1) if there is such that then for all This gives Hence,

[(d1) ⇒ (a1)]

Let be such that holds. There is such that Hence, and follows.

■

Lemma 2 : The following two statements are equivalent.

(a₂) [the non-trivial half of the theorem of the alternative]

for all α ϵ [-∞, +∞], $\sim I_\alpha \Rightarrow II_\alpha$

(d₂) there exists $y \in Y$ such that $g(y) \geq \inf_{x \in X} f(x)$

Proof : [(a₂) ⇒(d₂)]

If $\inf_X f = -\infty$ then clearly $g(y) \geq \inf_X f$ for each $y \in Y$. Now suppose $\inf_X f > -\infty$. Then $\sim I_\alpha$ holds for $\alpha = \inf_X f$. Therefore (a₂) implies that (II_α) holds. Thus there exists $y \in Y$ such that $g(y) \geq \alpha$ i.e. there exists $y \in Y$ such that $g(y) \geq \inf_X f = \alpha$. this is (d₂).

[(a₂) ⇒(d₂)]

Let (d₂) holds i.e., there exists $y \in Y$ such that $g(y) \geq \inf_X f$. Suppose $\alpha \in (-\infty, +\infty]$ be such that $\sim I_\alpha$ holds i.e., for all $x \in X, f(x) \geq \alpha$. This gives $\inf_X f \geq \alpha$. Then by (d₂), there is some $y \in Y$. such that $g(y) \geq \inf_X f \geq \alpha$.. This means (II_α) holds. ■

Proposition 3: (1) holds if and only if (4) holds.

Proof: The condition (1) is equivalent to (a₁) and (a₂) and (4) is equivalent to (d₁) and (d₂). Hence by previous two lemmas, we get (1) if and only if (4). ■

Tucker's Theorem 3.1: If B is a real skew-symmetric matrix there exists a non-negative vector u such that Bu is non-negative and $u + Bu$ is strictly positive.

Theorem 3.2

(AFFL: Alternative form of Farka's Lemma) :

IJournals International Journal of Software & Hardware Research in Engineering
ISSN No: 2347-4890 Volume 2 Issue 11, November-2014

Let $M \in R^{m \times n}$ and $c \in R^n$ be arbitrary. Then either

(A) There exists $v \geq 0$ such that $M^T v \leq c$ (i.e., $-M^T v + c \geq 0$)

or (B) There exists $w \geq 0$ such that $Mw \geq 0$ and $c^T w < 0$

but not both (A) and (B) hold.

Proof : Consider the following skew-symmetric matrix

$$B = \begin{bmatrix} 0 & 0 & M \\ 0 & 0 & -c^T \\ -M^T & c & 0 \end{bmatrix} \quad (10)$$

Now, by Tucker's Theorem 3.1, we get $u = \begin{bmatrix} v \\ t \\ w \end{bmatrix} \geq 0$, such that $Bu \geq 0$ and

$u + Bu > 0$. Now,

$$Bu = \begin{bmatrix} 0 & 0 & M \\ 0 & 0 & -c^T \\ -M^T & c & 0 \end{bmatrix} \begin{bmatrix} v \\ t \\ w \end{bmatrix} \geq 0 \quad (11)$$

implies, $Mw \geq 0$, $-c^T w \geq 0$, $-M^T v + ct \geq 0$. (12)

Then $u + Bu > 0$ gives $\begin{bmatrix} v \\ t \\ w \end{bmatrix} + \begin{bmatrix} Mw \\ -c^T w \\ -M^T v + ct \end{bmatrix} > 0$

implies, $v + Mw > 0$, $t - c^T w > 0$, $w - M^T v + ct > 0$. (13)

Case (i) : $t = 0$, (3) gives $Mw \geq 0$ and (4) gives $c^T w < 0$. Hence, (B) holds.

Case (ii) : $t = 1$, (3) gives $M^T v \leq c$. This is (A).

Further, if (A) and (B) hold together then by (A) $v^T M \leq c^T$ and by (B) there exists $w \geq 0$ such that $Mw \geq 0$ and $c^T w < 0$. Therefore, $v^T M w \leq c^T w < 0$.

But $Mw \geq 0$ and $v \geq 0$ gives $v^T M w \geq 0$. Hence, both (A) and (B) cannot exist together. ■

Theorem 3.3 : Farka's Lemma and AFFL are equivalent.

Proof : [⇒] According to Farka's Lemma, only one of the following system holds: For $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^n$,

(C) $Ax \leq 0$, $c^T x > 0$

(D) $A^T y = c$ and $y \geq 0$

In (c), replace x by $-x$ to get $Ax \geq 0$, $c^T x < 0$. This is the (B) of AFFL. Clearly, (D) implies (A) of the AFFL.

$[\Leftarrow]$ In (B) of AFFL, replace w by $-x$ to get (C). Now, apply AFFL to

$M = \begin{bmatrix} -A & A \end{bmatrix}$, $c^T = \begin{bmatrix} -b^T & b^T \end{bmatrix}$ to get (D). In this case $y = v$.

Notation : We denote by e the vector whose every element is 1 and by $|x|$ the vector whose elements are $|x_i|$.

Gales Theorem 3.4 : Either

(A_1) There exists $x \in \mathbb{R}^n$ such that $Ax \geq b$

or

(B_1) There exists $y \geq 0$ such that $A^T y = 0$ and $b^T y = 1$.

Proof : Let $M = \begin{bmatrix} A^T \\ -A^T \end{bmatrix}$, $v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $c = -b$ and $w = y$.

Substituting these values in (A) of AFFL, we get $M^T v \leq c$

implies, $\begin{bmatrix} A & -A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq -b$, i.e., $Ax_1 - Ax_2 \leq -b$. i.e., $Ax \geq b$ for $x = x_2 - x_1$.

This is (A_1).

Now, same substitutions in (A) and (B) of AFFL gives

$$\begin{bmatrix} A^T \\ -A^T \end{bmatrix} y \geq 0 \text{ and } -b^T y < 0$$

implies, $A^T y = 0$ and $b^T y > 0$.

Now, replace y by $\frac{y}{b^T y}$ to get $b^T y = 1$.

In the same way, we can show that various theorems of the alternative namely Gale, Gordan, Stiemke's, Motzkin and Dax are merely special cases of this AFFL and may be proved simply by substituting the appropriate matrices for *M*.

ACKNOWLEDGMENTS

With the blessing of my "Satguru" and the kind support of all professors of School of Mathematics, D. A. V. V., Indore, we have completed this paper. We are also thankful to our college authorities, friends and every one helped and encouraged ous.

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