*i*Journals: International Journal of Software & Hardware Research in Engineering (IJSHRE)
ISSN-2347-4890
Volume 8 Issue 10 October 2020

Proofs of Theorems Regarding Generating Functions

Sol Kim

Farragut High School

1. Abstract

In various fields, such as Discrete Mathematics, Combinatorics, and Number theory, number sequences often make an appearance in a problem. A generating function can be a powerful tool when working with them. It makes easier and shorter proofs of various relations when combinatorial proofs are way too long and complex. The idea behind the method is to treat a number sequence as a function and combinatorial formulas as algebraic relations between functions; an approach that appears to be very fruitful. There are different types of generating functions including ordinary, exponential, Dirichlet and others, which correspond to different types of recurrent relations. In the following, various theorems of different generating functions will be proved.

2. Introduction

In order to expand the polynomial $x+1$ raised to the n power, the Binomial Expansion Theorem can be used to easily express each term as follows:

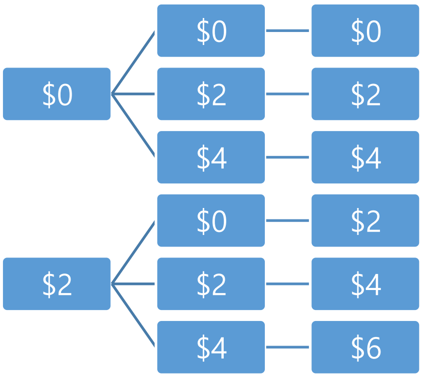
$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n$$

In this case, $(1+x)^n$ is the generating function and the coefficients of x^k represents the number of ways to choose k objects out of a total of n objects, such as committees of a certain amount of people.

With this in mind, counting combinations of objects via coefficients of polynomials can be applied to other problems. Two different example problems can demonstrate this concept.

Example 1: Suppose there are two spinners, A and B, from which a player can win money from if they spun each spinner once. Spinner A offers $0 or $2, and Spinner B offers $0, $2, or $4. What would be all of the combinations of money that could be earned, and how many ways could these amounts of money be earned?

This could be answered via a basic tree diagram with all of the possible outcomes. That way, all of the possible total amounts of money can be expressed, and the occurrence of each total can be counted.



From the diagram, it is clear that there is one way to get \$0 and \$6, while there are two ways to obtain \$2 and \$4. A different way to count these would be with generating functions, where each scenario from each spinner can be expressed in a polynomial. Applying similar logic from expanding a binomial raised to a certain power, the coefficients of each term can describe the number of occurrences for each amount of money, which is represented through the term's exponent. For example, the polynomial that would describe the possible states of Spinner A would be $1 + x^2$ because there is one occurrence for each amount of money, \$0 and \$2. If there were two possibilities of winning \$0, then the polynomial would be $2 + x^2$. It follows that the polynomial that describes the possibilities from Spinner B would be $1 + x^2 + x^4$. Then the problem arises: how can these two polynomials be utilized to express all the possible amounts of money and how many times they occur? Recall that the polynomial $1 + x$ expresses how many ways there are to pick committees out of a group of one person, and $(1 + x)^2$ or $(1 + x)(1 + x)$ expresses how many ways there are to pick committees out of a group of two people. This implies that the two polynomials for the two Spinners can be multiplied to find the possible combinations of money and the amount of ways they can be obtained. Thus, the polynomial $(1 + x^2)(1 + x^2 + x^4) = 1 + 2x^2 + 2x^4 + x^6$ expresses the desired combinations and occurrences, as the coefficients of the x^0 and the x^6 terms are 1, and the coefficients x^2 and the x^4 are 2.

Although this seems inconvenient for a problem with a small amount of combinations, a generating function becomes a powerful tool when dealing with a much larger number of combinations.

Example 2: There are 25 people who are going into a store, in which they all must spend money and buy one item each. One person can spend \$3, \$5, and \$9; and the other 24 can spend either \$1 or \$2. How many ways are there for this group of people to spend \$39?

Clearly, a tree diagram would be extremely inconvenient for this problem, as there would be 3×2^{24} branches. Combinatorics could be used to narrow this number down to find the answer, but it would be conceptually difficult and time-consuming. Instead, a generating function for each customer can be used. For the person who can spend \$3, \$5, or \$9, the generating function would be $x^3 + x^5 + x^9$ because there is one way for that person to spend any of the aforementioned amounts of money, which are represented by the exponents of each term. Then, the generating function for each of the other 24 people would be $x + x^2$. As observed earlier, the generating function for all the combinations of money would be the product of these polynomials, or $(x^3 + x^5 + x^9)(x + x^2)^{24}$. The final step is to count how many x^{39} terms there are. It should be noted that an x^{24} can be factored from the polynomial that represents the group of 24 people, which represents the fact that at least \$24 are going to be spent by that group of people. Now the generating function can be rewritten as $x^{24}(x^3 + x^5 + x^9)(1 + x)^{24}$. This problem can now be broken up into 3 cases, each representing how much the first person spends.

Case 1: First person spends \$3. If the first person spends \$3 and everyone else is guaranteed to spend at least \$24, then there are $39 - 24 - 3 = 12$ dollars left to spend, which will come from the polynomial $(1 + x)^{24}$. In order to get the term that represents \$12, the coefficient of the x^{12} term is needed, which would be $\binom{24}{12}$.

Case 2: First person spends \$5. Similar logic as the previous case can be used to find that there are $39 - 24 - 5 = 10$ dollars left to spend. Thus, the coefficient of the x^{10} term of $(1 + x)^{24}$ needs to be determined, which would be equal to $\binom{24}{10}$.

Case 3: First person spends \$9. Because there are $39 - 24 - 9 = 6$ dollars left to spend, the desired number is $\binom{24}{6}$.

Therefore, to finish this problem off, there are $\binom{24}{12} + \binom{24}{10} + \binom{24}{6}$ ways for this group of people to spend \$39.

The previous problems have essentially been asking the same question: *Given certain restrictions on $a_1, a_2, a_3 \ldots a_n$, how many ways can the sum of $S = a_1 + a_2 + a_3 + \ldots + a_n$ be achieved?*

However, certain variations on this problem have no restrictions on the terms $a_1, a_2, a_3 \ldots a_n$, such as the one as follows:

Example 3: How many ordered triplets (a, b, c) are there such that $a + b + c = 40$ and $a, b, c \geq 0$?

This type of problem would normally be solved using a technique known as stars and bars, where one would imagine a certain number of objects (which in this case is 40) and another number of dividers, where the space between two dividers would represent the value of one of the variables a, b, c. The number of dividers would be expressed as $k-1$, where $k$is the number of variables. Because each of the variables can have the value of 0, the expression that would give the number of ways to distribute n objects among k variables is$\binom{n+k-1}{k-1}$. Therefore, the number of solutions to the aforementioned equation is $\binom{40+3-1}{3-1} = \binom{42}{2} = 861$.

To solve this problem using generating functions, the first step is to develop polynomials that can represent the possible values ofthe three variables a, b, c. At first, one might think of the polynomial $1 + x + x^2 + \ldots + x^{40}$, since there is only one way for each of the variables to have the values of 0, 1, 2, . . . 40, which would produce the polynomial $(1 + x + x^2 + \ldots + x^{40})^3$ for the sum of the three variables, in which case the coefficient of the x^{40} term must be found. However, the more convenient method is to surprisingly let the polynomial for each variable to continue on forever because if the number of terms were to be restricted for each polynomial, the values of the coefficients would increase with each power of x and then decrease once it has hit a peak, which would make finding the coefficient of the x^{40} term more difficult. By letting each polynomial continue forever, the coefficients would strictly increase, which would actually make finding the desired coefficient simpler. This representation of each variable is actually more accurate since the problem states that each variable is any nonnegative integer. The first step would be to see what the pattern of the coefficients would be for $(1 + x + x^2 + \ldots)^2$ would be. It turns out that this polynomial would be $(1 + 2x + 3x^2 + \ldots)$, where the coefficients increase linearly. The next step is to multiply $(1 + 2x + 3x^2 + \ldots)(1 + x + x^2 + \ldots)$. The resulting polynomial would be $(1 + 3x + 6x^2 + 10x^3 + \ldots)$, where the coefficients are the triangular numbers, which could also be expressed via combinatorics as $\binom{n}{2}$, where n is some positive integer. The general form of the coefficient of the x^n term in this case would be $\binom{n+2}{2}$. It should be noted that this coefficient is equal to the number that was formulated via stars and bars, $\binom{n+k-1}{k-1}$, where$k = 3$ in this case. Thus, the answer using generating functions, $\binom{42}{2}$, is the same as the one using stars and bars.

There are many variations on this problem: a different number of variables, restrictions on what the variables can be, and having the variables sum to a different natural number. In the case of a different number of variables, the formula $\binom{n+k-1}{k-1}$ can be adapted to the case. For example, if there were 4 nonnegative integers that add to n, then there would be $\binom{n+4-1}{4-1} = \binom{n+3}{3}$ combinations of nonnegative integers to obtain the desired sum. This number would be the coefficient of the x^n of the polynomial $(1 + x + x^2 + \ldots)^4$.The reason this is can be attributed

*i*Journals: International Journal of Software & Hardware Research in Engineering (IJSHRE)
ISSN-2347-4890
Volume 8 Issue 10 October 2020

to hockey-stick identity, which is a method of visualizing the sum of combinatorial terms that are on Pascal's Triangle. Say that the coefficient of the x^4 term is desired from the polynomial $(1+x+x^2+...)^4$. It was previously stated that $(1+x+x^2+...)^3=(1+3x+6x^2+...)$, in which the coefficients can be expressed as $\binom{k+2}{2}$, where k is the exponent of the term. Therefore, the x^4 term would have the coefficient $\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\binom{5}{2}+\binom{6}{2}=\binom{2}{0}+\binom{3}{1}+\binom{4}{2}+\binom{5}{3}+\binom{6}{4}$. A path can be followed on Pascal's Triangle can be followed to determine the sum of these combinatorics.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

The red numbers are part of the aforementioned sum, while the blue number is the sum of the 5 combinatoric terms. All of these trace a hockey-stick pattern, hence the name of the identity. This also sheds light on why the coefficients of terms of $(1+x+x^2+x^3+...)^n$ follow a specific pattern.

These examples seem rather impractical using generating functions when there is already a counting technique that is more widely used; however, there are certain problems where there are restrictions placed on the variables that stars and bars would complicate the problem solving process greatly.

Example 4: There are 100 pieces of candy that are to be distributed among 5 friends. 2 of these friends can only have 0 or 1 piece of candy, 2 other friends can only have an odd number of pieces of candy, and the last one can have any nonnegative number of pieces. How many ways are there to distribute these pieces of candy?

As with the previous problems, polynomials that represent how many pieces of candy each person gets must be generated. The polynomial that represents the first two friends can be expressed as $1+x$, as there is only one way for each friend to receive either zero or one piece of candy. Similarly, the polynomial that represents the next two friends mentioned in the problem is $(x+x^3+x^5+...)$, where each of the exponents are odd numbers because they represent the number of pieces of candy that can be received. The final polynomial would be one that was examined in the previous example, $(1+x+x^2+...)$. Thus, the polynomial that represents the sum of the pieces of candy that each friend receives would be $(1+x)^2(x+x^3+x^5+...)^2(1+x+x^2+...)$. Now the real problem begins: how would the general form of the coefficients be expressed? Clearly, this would be a daunting task, as writing out $(x+x^3+x^5+...)^2$ term by term would be extremely inconvenient, and finding a pattern that is present throughout the coefficients would be difficult. On top of that, the new polynomial would have to be multiplied by the other infinite polynomial, which is another challenge in and of itself. It should be noted that the infinite polynomials are also infinite geometric series, which can be expressed as $\frac{a_0}{1-r}$, where a_0 is the first term of the geometric series and r is the common ratio between each term such that $r \in (-1, 1)$. This proposition begs the question: can this formula be used to condense the previously stated polynomials given the previously stated restriction? The answer is that it can be used, as the variable x is simply an indeterminate and can take any value that is assigned to it because only the exponent and coefficient of the terms are of interest, not the value of x itself. Therefore, the polynomial $(1+x)^2(x+x^3+x^5+...)^2(1+x+x^2+...)$ can be expressed as$(1+x)^2\left(\frac{x}{1-x^2}\right)^2(\frac{1}{1-x})$. This is much more convenient, as this can be simplified to $\frac{x^2}{(1-x)^3}$, which can be expanded into $x^2(1+x+x^2+...)^3$. All that is left is to find the coefficient of the x^{98} term, as there is already an x^2 term factored out of the polynomial. Using the general form of the coefficients $\binom{n+k-1}{k-1}$, where $n=98$ and $k=3$, it can be seen that given the restrictions on how many pieces of candy each of the friends can have, there are $\binom{98+3-1}{3-1}=\binom{100}{2}=4950$ ways to distribute 100 pieces of candy.

3. Defining Sequences

Before beginning the discussion regarding sequences, they must primarily be defined. The informal definition is the following: asequenceis an enumerated collection of objects in which repetitions are allowed. Some famous examples of sequences include infinite arithmetic progressions, infinite geometric progressions and the Fibonacci sequence, in which the members of each are indexed with integers. In other words, one can define a number sequence as a map from the set of integer indices to some set of numbers. Number sequences could be defined through various recurrence relations. For instance, the Fibonacci sequence$\{F_k\}_{k=0}^{\infty}$can be defined through the following recurrence relation:

$$F_{k+1}=F_k+F_{k-1}$$

with the first terms $F_0 = 0$ and $F_1 = 1$.Another way that the aforementioned sequence can be defined is through the use of the following series:

$$F_{k+1} = 1 + \sum_{i=0}^{k-1} F_i$$

4. Ordinary Generating Functions

The ordinary generating function of a sequence $\{a_k\}_{k=0}^{\infty}$ is defined as this formal series:

$$a(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$$

It should be noted that the word *formal* refers to the fact that convergence is not important in contrast to the evaluation of series in analysis. Despite this, formal series can be added and multiplied. In addition,the usage of the word *function* does not imply that this expression is a function; as mentioned earlier, the actual value of the series is meaningless, and the only aspects of the series that are of concern are the coefficients and exponents of each term.

Let $\{a_k\}_{k=0}^{\infty}$and $\{b_k\}_{k=0}^{\infty}$be defined as number sequences. If generating functions were to be defined for each sequence, then the sum of the two would also be a generating function. The generating functions for each respective sequence can be defined as

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$$

and

$$\sum_{k=0}^{\infty} b_k x^k = b_0 + b_1 x + b_2 x^2 + \cdots$$

When the series are added together, it should be noted that the coefficient of the term x^n would simply be $a_n + b_n$. Thus, the generating function that expresses the sum of the sequences $\{a_k\}_{k=0}^{\infty}$and $\{b_k\}_{k=0}^{\infty}$, $\{c_k\}_{k=0}^{\infty}$, can be expressed as:

$$\sum_{k=0}^{\infty} (a_k + b_k) x^k = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$$

Now to examine the product of the generating functions of the sequences $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$. The coefficients are not as simple as those of the sum of the two generating functions, and they may seem to lack an observable pattern. If these two generating functions are multiplied together, the following generating function for the sequence $\{d_k\}_{k=0}^{\infty}$ is obtained

$$\sum_{k=0}^{\infty} d_k x^k = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \cdots$$

However, this generating function can be expressed as a nested summation, which can be written as follows:

$$\sum_{k=0}^{\infty} d_k x^k = \sum_{k=0}^{\infty} (\sum_{j=0}^{k} a_{k-j} b_j) x^k$$

Some specific cases of the generating function for the sequence $\{d_k\}_{k=0}^{\infty}$ include the following examples:

Example 1: $b_0 = 1, b_k = 0 \; \forall \; k \geq 1$

Because the generating function for $\{d_k\}_{k=0}^{\infty}$ is defined as the product of the generating functions for $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$, the generating function $d(x)$ would b simply be $a(x) \times b(x)$. $b(x) = 1$, which would lead to the result that $d(x) = 1 \times a(x) = a(x)$.

Example 2: $b_m = 1, b_k = 0 \; \forall \; k \neq m$

As mentioned, the generating function for $\{d_k\}_{k=0}^{\infty}$ is the product of the generating functions for $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$. In this case, $b(x) = x^m$, and $a(x) = \sum_{k=0}^{\infty} a_k x^k$. This means that $d(x) = a(x) \times b(x)$ would equal the following

$$d(x) = x^m \sum_{k=0}^{\infty} a_k x^k$$

$$= \sum_{k=0}^{\infty} a_k x^{k+m}$$

In some cases, ordinary generating functions can be expressedina closed form. Here is one example to consider:

$$a_k = 1 \; \forall \; k \geq 0$$

Then the closed form for this generating function is

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots = \frac{1}{1-x}$$

This is the same statement that was stated when solving the example problems that involve distributing a certain number of objects among another number of categories.

Because of the previous statement, the following theorem for ordinary generating functions can be proven:

Given that

$$a(x) = \sum_{k=0}^{\infty} a_k x^k$$

and

$$d_k = \sum_{i=0}^{k} a_i$$

The generating function for $\{d_k\}_{k=0}^{\infty}$ can be expressed as

$$d(x) = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots$$

$$= a_0 + a_0 x + a_0 x^2 + \cdots + a_1 + a_1 x + a_1 x^2 + \cdots$$

$$= a_0 \sum_{i=0}^{\infty} x^i + a_1 \sum_{i=0}^{\infty} x^{i+1} + a_2 \sum_{i=0}^{\infty} x^{i+2} + \cdots$$

$$= a_0 \left(\frac{1}{1-x}\right) + a_1 x \left(\frac{1}{1-x}\right) + a_2 x^2 \left(\frac{1}{1-x}\right) + \cdots$$

$$= (\frac{1}{1-x})(a_0 + a_1 x + a_2 x^2 + \cdots)$$

$$= \frac{a(x)}{1-x}$$

5. Generating Function with Multiple Indexes

In the previous section, generating functions were considered, which corresponds to general number sequences. However, the family of generating functions is much broader.

Sometimes, one may encounter sequences with multiple indexes. For example, the binomial coefficients

$$a_{n,k} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

are always indexed with two numbers n and k. For the collection of said numbers, the following ordinary generating function can be formatted as follows:

$$a(x,y) = \sum_{n,k=0}^{\infty} a_{k,n} x^k y^{n-k}$$

In this example, the number of variables in the formal series is equal to the number of indexes which were used in the enumeration. If there are two variables like in the example above, then it is called bivariate generating function.

As mentioned, by the Binomial Theorem, the polynomial $(x+y)^n$ can be expanded to be the following:

$$(x+y)^n = \binom{n}{n} x^n + \binom{n}{n-1} x^{n-1} y + \cdots + \binom{n}{0} y^n = \sum_{k=0}^{n} \binom{n}{n-k} x^{n-k} y^k$$

Consider the sum of these polynomials for all nonnegative values of n. Then the sum can be rewritten as the following:

$$\sum_{n=0}^{\infty} (x+y)^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{n-k} x^{n-k} y^k$$

The aforementioned sum can also be written in a closed form as follows

$$\sum_{n=0}^{\infty} (x+y)^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{n-k} x^{n-k} y^k = \frac{1}{(1-x-y)}$$

6. Exponential Generating Functions and Operations

Another type of generating functions is the exponential generating function, and it can be written as follows:

$$a(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!} = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + \cdots$$

The $\frac{x^k}{k!}$ terms may look familiar, and that is because the generating function is derived from the Maclaurin series for the function $f(x) = e^x$; hence the name, exponential generating function. The Maclaurin series for $f(x) = e^x$ is written as the following:

$$e^x = 1 + x + \frac{x^2}{2!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

As for operations on exponential generating functions, addition of two of these functions, $a(x)$ and $b(x)$seems rather self-explanatory: the coefficients of each term would look something like $(a_k + b_k)\frac{x^k}{k!}$. However, multiplication is slightly more complicated, as it may not be clear as to what the general form for a term may be. In order to find this general form, the two exponential generating functions, $a(x)$ and $b(x)$, can be multiplied manually.

$$\mathrm{a(x)} \cdot \mathrm{b(x)} = \left(\sum_{k=0}^{\infty} a_k \frac{x^k}{k!}\right)\left(\sum_{k=0}^{\infty} b_k \frac{x^k}{k!}\right)$$

$$= \left(a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + \cdots\right)\left(b_0 + b_1\frac{x}{1!} + b_2\frac{x^2}{2!} + \cdots\right)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \left(\frac{a_0 b_2}{2!} + a_1 b_1 + \frac{a_2 b_0}{2!}\right)x^2$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{a_{m-k} b_k}{(m-k)!\,k!} x^m$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{a_{m-k} b_k m!\,x^m}{(m-k)!\,k!\,m!}$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{m} a_{m-k} b_k \binom{m}{k}\frac{x^m}{m!}$$

The significance of choosing terms to take the form of $\frac{x^k}{k!}$ may not seem clear, as there seem to be no advantages in choosing to make a sequence of numbers an exponential generating function versus an ordinary one. However, one significant advantage becomes clear when two new operation are introduced to generating functions: differentiation and integration.

Because generating functions are simply polynomials, no advanced differentiating or integrating techniques are needed. The only principle that is required to perform these operations on generating functions is the Power Rule. Taking the derivative and integral of some polynomial function x^n where $n \in \mathbb{Z}$ can be explained as the following:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\int x^n\,dx = \frac{x^{n+1}}{(n+1)}$$

Now these operations can be performed on the exponential generating function,

$$a(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!} = a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + \cdots$$

The derivative of the generating function $a(x)$ is

$$\frac{d}{dx}a(x) = a_1 + a_2x + a_3\frac{x^2}{2!} + \cdots = \sum_{m=0}^{\infty} a_{m+1}\frac{x^m}{m!}$$

Differentiation shifts every term of the sequence $\{a_k\}_{k=0}^{\infty}$ by one term. Similarly, integration does the same but in a different direction.

$$\int a(x)dx = a_0x + a_1\frac{x^2}{2!} + a_2\frac{x^3}{3!} + \cdots = \sum_{m=1}^{\infty} a_{m-1}\frac{x^m}{m!}$$

It should be noted that the a_0 term disappears when differentiating the generating function $a(x)$, while there is a lack of a constant term when integrating $a(x)$. Technically, by definition of an indefinite integral, there should be a constant term after integrating $a(x)$, but there is no defined term from the sequence $\{a_k\}_{k=0}^{\infty}$ to take that position.

Now that these operations have been introduced, the differentiation and integration of ordinary generating functions should also be considered. $b(x)$ will be defined as an ordinary generating function for the sequence $\{b_k\}_{k=0}^{\infty}$ and will take the form of

$$b(x) = \sum_{k=0}^{\infty} b_kx^k$$

If $b(x)$ were to be differentiated, it would take the following form:

$$\frac{d}{dx}\big(b(x)\big) = b_1 + 2b_2x + 3b_3x^2 + \cdots = \sum_{k=0}^{\infty}(k+1)b_{k+1}x^k$$

Additionally, if $b(x)$ were to be integrated, it would be written as follows:

$$\int b(x)dx = b_0x + b_1\frac{x^2}{2} + b_2\frac{x^3}{3} + \cdots = \sum_{k=0}^{\infty}\frac{b_k}{(k+1)}x^{k+1}$$

Compared to differentiating and integrating the previously defined exponential generating function $a(x)$, it can be seen that performing either of these two terms has complicated each term of $b(x)$ by adding another rational coefficient to each term. Put shortly, when differentiating or integrating exponential functions, it simply shifts the terms of the sequence $\{a_k\}_{k=0}^{\infty}$ to another $\frac{x^k}{k!}$ term, while performing either of these operation on an ordinary generating function such as $b(x)$ will complicate each term of the function while shifting the terms.

7. Proofs of Theorems with Exponential Generating Functions

The first example of exponential generating function is the sequence with all coefficients equal to one.

$$\sum_{m=0}^{\infty} a_m \frac{x^m}{m!} = \sum_{m=0}^{\infty} 1 \cdot \frac{x^m}{m!}$$

One can recognize this expression as the Maclaurin series of the exponential function as mentioned in previous chapter, and its closed form is defined to be

$$e^x = \sum_{m=0}^{\infty} 1 \cdot \frac{x^m}{m!}$$

The following combinatorial theorems can be proven thanks to this closed form:

Theorem 1:

$$\sum_{k=0}^{m} \binom{m}{k} = 2^k$$

Method 1: This is a classic combinatorial proof. $\binom{m}{k}$ is essentially the number of ways to pick m elements out of a set of size m. If these values are summed from $k = 0$ to $k = m$, then it represents all the subsets with m elements. Thus, it can be said that each element is either included of excluded from the set. Therefore, the expression that would represent this would be 2^m, as 2 represents the state of each element.

Method 2: For this proof, the generating function will be utilized. Consider the Maclaurin series for e^{2x}. The series for this expression would be

$$e^{2x} = \sum_{m=0}^{\infty} \frac{(2x)^m}{m!} = \sum_{m=0}^{\infty} 2^m \frac{(x)^m}{m!}$$

On the other hand, another way to express e^{2x} would be $e^x \cdot e^x$. The multiplication of two exponential generating functions has already been discussed above. From that closed form, the following can be stated:

$$e^x \cdot e^x = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \binom{m}{k} \frac{x^m}{m!}$$

By substitution, these summations can be set equal, and it can be inferred that

$$\sum_{k=0}^{m} \binom{m}{k} = 2^m$$

Theorem 2

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} = 0$$

Proof

Method 1: Combinatorically, this may be difficult to formulate. Consider flipping n coins. This sum is essentially equal to the number of ways to flip an even number of heads out of n coins minus the number of ways to flip an odd number of heads out n coins. Whether the number of coins flipped is even or odd relies on the last coin flipped. Since heads and tails are equally likely, the sum must evaluate to 0.

Method 2: A similar product will be considered as the one above using generating functions. It is known that $e^0 = 1$. Using the Maclaurin series, it can be said that

$$e^0 = \sum_{m=0}^{\infty} 0 \cdot \frac{x^m}{m!}$$

The equivalent product would be between $e^x \cdot e^{-x}$. The generating function for this would be

$$e^x \cdot e^{-x} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{x^m}{m!}$$

By similar logic as the last proof,

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} = 0$$

for the coefficients to match.

8. Fibonacci Generating Functions

In this section, the well-known Fibonacci Sequence will be discussed. In order to study their properties, an ordinary generating function will be used.

The ordinary generating function for the Fibonacci numbers is as follows:

$$f(x) = \sum_{n=0}^{\infty} F_n x^n$$

$$F_n = F_{n-1} + F_{n-2} \; for \; n \geq 3, and \; F_0 = 0, F_1 = 1$$

$$F_{n+2} \cdot x^{n+2} = (F_{n+1} + F_n) \cdot x^{n+2}$$

$$F_{n+2} \cdot x^{n+2} = F_{n+1} \cdot x^{n+2} + F_n \cdot x^{n+2}$$

$$\sum_{n=0}^{\infty} F_{n+2} \cdot x^{n+2} = \sum_{n=0}^{\infty} F_{n+1} \cdot x^{n+2} + \sum_{n=0}^{\infty} F_n \cdot x^{n+2}$$

$$f(x) - F_1 x - F_0 = x(f(x) - F_0) + x^2 f(x)$$

$$f(x) - x = xf(x) + x^2 f(x)$$

$$f(x)(1 - x - x^2) = x$$

$$f(x) = \frac{x}{1 - x - x^2}$$

This is the closed form of the ordinary generating function for the Fibonacci numbers. Using this closed form of the Fibonacci sequence, the following recursive formula can be proven:

$$F_{k+1} = 1 + \sum_{i=0}^{k-1} F_i$$

Proof:

Consider the set S such that each element is given by

$$s_k = \sum_{n=0}^{k} F_n$$

If the formula were assumed to be true, then the generating function for s_k would be

$$s(x) = \frac{f(x)}{1 - x}$$

Going along the assumption that the formula is true, then

$$F_{k+2} = 1 + \sum_{i=0}^{k} F_i = 1 + s_k$$

$$F_{k+2} \cdot x^{k+2} = x^{k+2} + s_k \cdot x^{k+2}$$

$$\sum_{k=0}^{\infty} F_{k+2} \cdot x^{k+2} = \sum_{k=0}^{\infty} x^{k+2} + \sum_{k=0}^{\infty} s_k \cdot x^{k+2}$$

$$f(x) - F_1 x - F_0 = \frac{x^2}{1-x} + s(x) \cdot x^2$$

$$f(x) - x = \frac{x^2}{1-x} + s(x) \cdot x^2$$

$$f(x) - x = \frac{x^2 + f(x) \cdot x^2}{1-x}$$

$$f(x)\left(1 - \frac{x^2}{1-x}\right) = \frac{x^2}{1-x} + x$$

$$f(x)\left(\frac{1-x-x^2}{1-x}\right) = \frac{x}{1-x}$$

$$f(x) = \frac{x}{1-x-x^2}$$

By assuming that the sum was true, the closed form of the generating function for the Fibonacci sequence has been found, which is known to be true. Therefore, the sum must also be true.

There is another interest combinatorial theorem that can be derived from the Fibonacci generating function

Theorem 2

$$F_n = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-k-1}{k}$$

Proof:

The proof can be started by considering the closed form of

$$\sum_{n=0}^{\infty} (x+y)^n$$

which would be equal to

$$\frac{1}{1-x-y}$$

However, the sum can also be written as a double summation, which introduces a desired combinatorial term

$$\sum_{n=0}^{\infty} (x+y)^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

If x were to be replaced with y^2, this formula would look similar to the closed form of the Fibonacci generating function

$$\frac{1}{1-y-y^2} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} y^{n-k} y^{2k}$$

The indices of each term can then be changed:

$$\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\binom{n-1}{k}y^{n-k-1}y^{2k}$$

If the first few terms of the expansion of this sum were to be inspected closely, one might notice that

$$\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\binom{n-1}{k}y^{n-k-1}y^{2k} = \sum_{n=1}^{\infty}\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-k-1}{k}y^{n-2k-1}y^{2k}$$

This new equivalent form can then be simplified to

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-k-1}{k}y^{n-2k-1}y^{2k} = \sum_{n=1}^{\infty}\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-k-1}{k}y^{n-1}$$

It should be recalled that the sum was remarkably close to the closed form of the Fibonacci generating function:

$$\frac{1}{1-y-y^2} = \sum_{n=1}^{\infty}\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-k-1}{k}y^{n-1}$$

By multiplying both sides by y,

$$\frac{y}{1-y-y^2} = \sum_{n=1}^{\infty}\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-k-1}{k}y^{n}$$

Finally, by recalling that

$$\frac{y}{1-y-y^2} = \sum_{n=1}^{\infty}F_n y^n$$

The coefficients of the sums can be compared:

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-k-1}{k}y^{n} = \sum_{n=1}^{\infty}F_n y^n$$

Thus, it can be seen that

$$\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-k-1}{k} = F_n$$

9. Dirichlet Series

The final and perhaps most important generating function that has not been investigated in this paper is the Dirichlet Series. The Dirichlet Series takes the form of

$$A(x) = \sum_{n=1}^{\infty}\frac{a_n}{n^x}$$

In order to inspect some theorems relating to this type of series, a_n will $n \geq 1$. Allow this generating function to be $Z(x)$

$$Z(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

This may seem very familiar, and that is because it is the infamous Riemann Zeta Function.

Just like the other generating functions, the product of two different Dirichlet Generating Functions

$$a(x) = \sum_{n=1}^{\infty} \frac{a_n}{n^x}, b(x) = \sum_{n=1}^{\infty} \frac{b_n}{n^x}$$

$$a(x) \cdot b(x) = (\frac{a_1}{1^x} + \frac{a_2}{2^x} + \frac{a_3}{3^x} + \cdots)(\frac{b_1}{1^x} + \frac{b_2}{2^x} + \frac{b_3}{3^x} + \cdots)$$

$$= (\frac{a_1 b_1}{1^x 1^x} + \frac{a_1 b_2}{1^x 2^x} + \frac{a_2 b_1}{1^x 2^x} + \frac{a_1 b_3}{1^x 3^x} + \cdots)$$

$$\sum_{n=1}^{\infty} \sum_{d|n} \frac{a_d b_{\frac{n}{d}}}{n^x}$$

where $d|n$ represents all divisors of n.

An example of this would be the square of the Riemann Zeta function. The result of this would include $d(n)$, or the number of divisors of n. If $Z(x)$ is the Riemann Zeta function, then

$$\left(Z(x)\right)^2 = Z(x) \cdot Z(x)$$

$$= (\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \frac{1}{5^x} + \frac{1}{6^x} + \cdots)(\frac{1}{1^x} + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \frac{1}{5^x} + \frac{1}{6^x} + \cdots)$$

$$= \frac{1}{1^x} + \frac{2}{2^x} + \frac{2}{3^x} + \frac{3}{4^x} + \frac{2}{5^x} + \frac{4}{6^x} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{d(n)}{n^x}$$

Another interesting theorem that is related to the Riemann Zeta function is the following:

Theorem 1 Euler Product Theorem

$$\prod_p \left(1 - \frac{1}{p^x}\right)^{-1} = Z(x)$$

where p is a prime

Proof

$$\prod_p \left(1 - \frac{1}{p^x}\right)^{-1} = \prod_p \frac{p^x}{p^x - 1}$$

$$= \prod_p \frac{1}{1 - p^{-x}}$$

$$= \prod_p \sum_{n=0}^{\infty} p^{-nx}$$

$$= \left(1 + \frac{1}{2^x} + \frac{1}{4^x} + \cdots\right)\left(1 + \frac{1}{3^x} + \frac{1}{9^x} + \cdots\right)\left(1 + \frac{1}{5^x} + \frac{1}{25^x} + \cdots\right)\cdots$$

$$= 1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \cdots$$

$$= Z(x)$$

Another theorem that relates to the Zeta function involves the Euler Totient Function. The function is denoted by $\varphi(x)$, and it represents the number of positive integers less than or equal to x that are relatively prime to x. For example, $\varphi(8) = 4$, since the positive integers that are less than or equal to 8 that are relatively prime to it are 1, 3, 5, and 7. One crucial property of $\varphi(x)$ is that it is multiplicative. Through elementary counting methods, several formulae can be determined to calculate $\varphi(p^n)$ for a prime p and an integer n.

Firstly, $\varphi(p)$ would be equal to $p - 1$, since all positive integers less than p are relatively prime to p. To calculate $\varphi(p^n)$, the number of positive integers less than or equal to p^n that are not relatively prime to p^n would be any multiple of p. There would be $\frac{p^n}{p} = p^{n-1}$ of these, and these would simply have to be excluded from the number of positive integers up to p^n to become $p^n - p^{n-1}$. Using the multiplicativity of this function, $\varphi(x)$ can be found for any integer because of the unique factorization theorem of integers.

By the Principle of Inclusion and Exclusion, an alternate definition of $\varphi(n)$ would be

$$\varphi(n) = n \prod_{p|n} (1 - \frac{1}{p})$$

such that p is a prime.

Another theorem regarding the Euler Totient Function is the following:

$$\sum_{d|n} \varphi(d) = \sum_{d|n} \varphi(\frac{n}{d}) = n$$

This theorem will be used for the proof of the next theorem

Theorem 2

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^x} = \frac{Z(x-1)}{Z(x)}$$

The final theorem that will be inspected is the one above. This essentially shows what the Dirichlet series for the Euler Totient Function is equal to.

Proof

Consider the multiplication of the Dirichlet Series of the Euler Totient function by the Riemann

Zeta function

$$Z(x) \cdot \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^x}$$

Note that the product formula of two Dirichlet Series can be used to simplify the expression

$$Z(x) \cdot \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^x} = (\sum_{n=1}^{\infty} \frac{1}{n^x}) \cdot (\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^x})$$

$$= \sum_{n=1}^{\infty} \sum_{d|n} \frac{\varphi(\frac{n}{d})}{n^x}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^x} \sum_{d|n} \varphi(\frac{n}{d})$$

The aforementioned divisor sum of the Totient Function will be used to obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^x} \sum_{d|n} \varphi(\frac{n}{d}) = \sum_{n=1}^{\infty} \frac{1}{n^x} \cdot n$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{x-1}}$$

Finally, this summation can be simplified to $Z(x-1)$. So, the equation, after simplification, is the following:

$$Z(x) \cdot \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^x} = Z(x-1)$$

Dividing both sides by $Z(x)$, the theorem is obtained

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^x} = \frac{Z(x-1)}{Z(x)}$$

10. Summary

In conclusion, theorems regarding ordinary, exponential, Fibonacci, and Dirichlet generating functions were proven, including closed forms of series, combinatorial identities, and identities in number theory. From these generating functions, new perspectives of fields of mathematics can be seen. The scope of generating functions extends far past what has been discussed, including statistics and cryptography.Despite what is already known, there are still mysteries surrounding the behavior of certain functions, an example being the Riemann Zeta function.

