

UTILIZATION ABAOUB-SHKHEAM TRANSFORM IN SOLVING LINEAR INTEGRAL EQUATIONS OF VOLTERRA

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ABSTRACT

In this work, we show the theorem of the convolution of the Abaoub-Shkheam transform, and employed it for solving one dimensional linear Volterra integral equations of the second kind.

Keywords: *The Abaoub-Shkheam transform, linear Volterra integral equations of the second kind*

1. INTRODUCTION

The Volterra integral equations play a vital role (like in population dynamics, the spread of epidemics, and semi-conductor devices, biological population, etc.). There are so many different techniques to solve Volterra integral equations; the integral transforms were extensively used. Recently, Abaoub-Shkheam introduced a new integral transform, named the Q -transform. The general form of Volterra integral equation is

$$v(x) - h(x) = \lambda \int_0^x g(x,t)v(t)dt, \quad (1)$$

$$Q[h(t)] = \int_0^\infty h(ut) e^{-\frac{t}{s}} dt = T(u,s), \quad t \geq 0, \quad (2)$$

where Q is the Abaoub-Shkheam operator.

The sufficient conditions for the existence of the Abaoub-Shkheam transform of the function $h(t)$ for $t \geq 0$ are the function $h(t)$ is piecewise continuous in every finite interval and of exponential order.

2. SOME PROPERTIES OF ABAOUB - SHKHEAM TRANSFORM

In this part, we present some properties of the Abaoub-Shkheam transform.

2.1. Linearity of Abaoub-Shkheam transform [1].

Theorem 1: Let $f(t)$ and $g(t)$ are functions, then for any constants a and b we have

$$Q[af(t) + bg(t)] = aQ\{f(t)\} + bQ\{g(t)\}$$

2.2. Abaoub-Shkheam transform of elementary functions [1]

We show in a table (1) the Abaoub-Shkheam transform for standard functions for $n \geq 0$.

Table 1. Abaoub-Shkheam transform for standered functions

$f(t)$	t^n	e^{at}	$\sin at$	$\cos at$
$Q[(f(t))]$	$n! u^n s^{n+1}$	$\frac{s}{1 - a u s}$	$\frac{a u s^2}{1 + a^2 u^2 s^2}$	$\frac{s}{1 + a^2 u^2 s^2}$

2.3. Inverse Abaoub-Shkheam transform:

If $Q[h(t)] = T(u, s)$, then the original function $h(t)$ is called the inverse of the Abaoub-Shkheam transform or inverse of $T(u, s)$, and is defined by

$$h(t) = Q^{-1}\{T(u,s)\}$$

Table (2).The inverse of the Abaoub-Shkheam transform for some elementary functions.

$T(u, s)$	$u^n s^{n+1}$	$\frac{s}{1 - a u s}$	$\frac{u s^2}{1 + a^2 u^2 s^2}$	$\frac{s}{1 + a^2 u^2 s^2}$
$Q^{-1}\{T(u, s)\}$	$\frac{t^n}{n!}$	e^{at}	$\frac{1}{a} \sin at$	$\cos at$

3. LAPLACE-Q DUALITY PROPERTY:

If the Laplace transform of the function $f(t)$ is $F(s)$, then

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt. \quad (3)$$

Substitute $t = vy$ in the integral on the right-hand side we get

$$F(s) = \mathcal{L}[f(t)] = u \int_0^{\infty} f(uy) e^{-suy} dy$$

If we substitute $vy = y$, then equation (2) becomes,

$$Q[f(t)] = \frac{1}{u} \int_0^{\infty} f(y) e^{-\frac{1}{us}y} dy = T(u, s) \quad (4)$$

Hence, from equation (4) we get

$$F(s) = u T\left(u, \frac{1}{us}\right), \quad (5)$$

also, from equations (2) and (3) we get

$$T(u, s) = \frac{1}{u} F\left(\frac{1}{us}\right). \quad (6)$$

The equations (5) and (6) form the duality relation between these two transforms and may serve as a means to get one from the other when needed.

4. THE CONVOLUTION OF TWO FUNCTIONS [1]:

The Convolution of two functions $f(t)$ and $g(t)$ is denoted by $f(t) * g(t)$ and it is defined by

$$f(t) * g(t) = f * g = \int_0^t f(x) g(t-x) dx = \int_0^t f(t-x) g(x) dx$$

4.1. Convolution theorem for Abaoub

- Shkheam transform:

Now we present the new result in this paper

Theorem 2: Let $f(t)$ and $g(t)$ be functions having Laplace transforms $F(s)$ and $G(s)$ respectively, and having the Abaoub-Shkheam transform $T_1(u, s)$ and $T_2(u, s)$ respectively. The Abaoub-Shkheam transform of the convolution of $f(t)$ and $g(t)$ is given by

$$Q[f(t) * g(t)] = u T_1(u, s) T_2(u, s)$$

Proof:

We know

$$\mathcal{L}[f(t) * g(t)] = F(s) G(s) = H(s).$$

If $Q[f(t) * g(t)] = T(u, s)$ then

$$T(u, s) = \frac{1}{u} H\left(\frac{1}{us}\right).$$

Hence

$$Q[f(t) * g(t)] = \frac{1}{u} F\left(\frac{1}{us}\right) G\left(\frac{1}{us}\right) = u T_1(u, s) T_2(u, s) \quad \square$$

5. THE ABAOUB-SHKHEAM TRANSFORM FOR SOLVING LINEAR VOLTE- RRA INTEGRAL EQUATIONS OF THE SECOND KIND

Here, we use the transform of the Abaoub-Shkheam for solving linear Volterra integral equations of the second kind given by (1). In this part, we will suppose that the kernel $g(x, t)$ of (1) is a difference kernel that can be expressed by difference $(x - t)$. Thus, equation (1) can be expressed as

$$v(x) - h(x) = \lambda \int_0^x g(x-t)v(t) dt \quad (7)$$

Applying the Abaoub-Shkheam transform on (7), we get

$$Q[v(x)] - Q[h(x)] = \lambda Q \left[\int_0^x g(x-t)v(t) dt \right].$$

By theorem (2), we obtain

$$Q[v(x)] - Q[h(x)] = \lambda u Q[g(x)] Q[v(x)]$$

$$Q[v(x)] = \frac{Q[h(x)]}{1 - \lambda u Q[g(x)]}. \quad (8)$$

Utilize the inverse of the Abaoub-Shkheam transform on (8), we get

$$v(x) = Q^{-1} \left\{ \frac{Q[h(x)]}{1 - \lambda u Q[g(x)]} \right\}.$$

This is the required solution to (7).

6. Examples

Now, we give some examples to prove the success

of the Abaoub-Shkheam transform to solve equations of Volterra.

Example 1.

Let

$$v(x) = \frac{x^3}{3!} + \int_0^x (t-x)v(t) dt \tag{9}$$

Applying the Abaoub-Shkheam transform on to both sides of (9), and then using the convolution theorem, we have

$$Q[v(x)] = \frac{t^3 s^4}{1+t^2 s^2} = t s^2 - \frac{t s^2}{1+t^2 s^2} \tag{10}$$

Taking the inverse of the Abaoub-Shkheam transform on (10), we obtain

$$v(x) = x - \sin x$$

Example 2.

Consider

$$v(x) = \cos x + \sin x - 2 \int_0^x \sin(t-x)v(t) dt \tag{11}$$

Taking the Abaoub-Shkheam transform on (11), and then using the convolution theorem, we have

$$Q[v(x)] = \frac{s + t s^2}{1 - t^2 s^2} = \frac{s}{1 - t s} \tag{12}$$

Operating inverse the Abaoub-Shkheam transform on (12), we obtain

$$v(x) = e^x$$

7. CONCLUSION

In this paper, we have successfully proved the convolution theorem of the Abaoub-Shkheam transform, and used it to solve equations of Volterra. The given examples proved that the exact solution has been gotten applying for a little colocation works and spending little time. The suggested scheme can be used for other equations of Volterra and their system.

8. REFERENCES

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