*i*Journals: International Journal of Software & Hardware Research in Engineering (IJSHRE)
ISSN-2347-4890
Volume 10 Issue 10 October 2022

A New Look At Groupoid And Cancellation Law.

Gambo, Jeremiah Gyam[1]; Tella, Yohanna[2]
ORCID: 0000-0002-6295-9802[1]; ORCID: 0000-0002-1642-6554[2]
Nassarawa State University, Keffi, Nigeria[1]; Kaduna State University, Kaduna, Nigeria[2]
E-mail: ggjerry016@gmail.com[1]; yot2002@yahoo.com[2]

ABSRACT. In this paper, the concepts of Multiset Groupoid is introduced. The condition for a submultiset of a multiset groupoid to be a submultiset groupoid is established and a study of the closure of multiset operations on class of finite multiset groupoids is carried out. It is also established that the root set of a multiset groupoid is a sub groupoid and a submultiset groupoid. Furthermore, the cancellability of an element of a multiset groupoid and that of multiset groupoid also defined. A necessary and sufficient condition for cancellability of a multiset groupoid is established. Homomorphism of multiset groupoids defined. The cancellability of the homomorphic image of a cancellable element and that of a cancellable multiset groupoid is determined . Commutative multiset groupoid has been shown regular whose root set is a commutative multiset groupoid

KEY WORDS: G**roupoid, Multiset, Submultiset, Root set, Multiset operations, Multiset groupoid, Cancellability, Commutativity and Homomorphism of multiset groupoids.**

1. Introduction.

A multiset (mset for short) is a collection of objects, unlike a standard Cantorian set (in which the elements are not allowed to repeat), here repetitions are allowed. For the various applications of multiset the reader is referred to article [18]. It is observed from the survey of available literature on msets and applications that the idea of mset was hinted by R. Dedekind in 1888. The mset theory which generalizes set theory as a special case was introduced by Cerf et al.[2]. The term mset, as noted by Knuth [4] was first suggested by N.G de Bruijn in a private communication to him. Further studies were carried out by Yager [14], Blizard [1]. Other researchers ([5], [7], [8]) gave a new dimension to the theory of msets.

From a practical point of view msets are very useful structures arising in many areas of mathematics and computer science [18]. Mset Topological space has been studied by Shravan and Tripathy [10]. Research on the theory of msets has been gaining grounds. The research carried out so far shows a strong analogy in the behaviour of sets and msets. It is possible to extend some of the main notion and results of sets to the setting of msets. In 2009, Girish and Sunil [3], introduced the concepts of relations, function, composition, and equivalence in msets context. Tella and Daniel ([12], [13]) have considered sets of mappings between msets and studied symmetric groups under mset perspective. Gambo and Tella [17] presented mset functions from a very unique and different way. It is build on the studies from previous research works. Some of the basic principles and properties of functions are studied in multiset context such as injection, surjection, bijection, identity, and constant functions. The composition of functions is studied. Similarity and dominance relations are also studied among others. Nazmul et al. [6] improved on Tella and Daniel's work and added two axioms which marks the foundations of studying group theory in mset perspective. Groupoid is a classical algebraic structure, though having numerous kind of definitions from the literatures in the classical sense. We adhered to the fundamental definition of groupoid in ([15] and [16]).

In this paper we present the study of mset groupoid referred to as multi groupoid (mgroupoid for short). In addition to this section, we present some preliminary definitions, basic notations and some existing results in the literature in section two to make the paper self-contained. In section three, the closure of mset operations on the class of finite mgroupoids are studied and cancellability of an element of an mgroupoid and that of an mgroupoid defined. Homomorphism of mset groupoids is defined and also studied. The entire paper is summarised in section four.

2. Preliminaries

2.1 Definitions and notations

Definition 2.1.1[15,16]: Let S be a set and $\mu: S \times S \to S$ a binary operation that maps each ordered pair (x, y) of S to an element $\mu(x, y)$ of S. The pair (S, μ)(or just S, if there is no fear of confusion) is called a *groupoid.* The mapping μ is called the product of (S, μ). We shall mostly write simply xy Instead of (x, y). If we want to emphasize the place of the operation then we often write $x.y$. The element $xy(= \mu(x, y))$ is the product of x and y in S. A Subset A of a groupoid (S, μ) is said to be a subgroupoid if and only if it is a groupoid under an induced operation μ.

Definition 2.1.2[1]: An mset A over the set X can be defined as a function $C_A: X \to \mathbb{N} = \{0,1,2, \dots\}$ where the value $C_A(x)$ denote the number of times or multiplicity or count function of x $in\ A$. For example, Let $A = [x, x, x, y, y, y, z, z]$, then $C_A(x) = 3, C_A(y) = 3, C_A(z) = 2$. $[C_A(x) = 0 \Leftrightarrow x \notin A]$. The mset M over the set X is said to be empty if $C_M(x) = 0\ for\ all\ x \in X$. We denote the empty mset by $\emptyset$. Then $C_\emptyset(x) = 0, \forall\ x \in X$. if $C_A(x) > 0, then\ x \in A$. Note that $x \in A \Leftrightarrow C_A(x) > 0$.
If $C_A(x) = n$ then the membership of x in A can be denoted by $x \in^n A$, meaning x belong to A exactly n times.

Definition 2.1.3[1]: The cardinality of a mset M denoted $|M|$ or $card(M)$ is the sum of all the multiplicities of its elements given by the expression $|M| = \sum_{x \in X} c_A(x)$.
An mset M over a set X is said to be finite if $|M| < \infty$. We denote the class of all finite msets over X by $M(X)$.

Note: Presentation of mset on paper work became a challenged as every researcher has his thought in that aspect . However the use of square brackets was adopted in ([1], [9],[11]) to represent an mset and ever since then it has become a standard. For example if the multiplicity of the elements x, y and z in an mset M are 2,3 and 2 respectively, then the mset M can be represented as $M = [x, x, y, y, y, z, z,]$, others may put it like $[x, y, z]_{2,3,2}$ or $[x^2, y^3, z^2]$ or $[x2, y3, z2]\ or\ [2/x, 3/y, 2/z]$ depending on one's taste or expediencies. But for conveniences sake, curly bracket can be used instead of the square bracket.

Definition 2.1.4[2]: Let M be an mset drawn from a set X. The support set of M denoted by M^* is a subset of X given by $M^* = \{x \epsilon X: C_M(x) > 0\}$. M^* is also called root (support) set.

Definition 2.1.5[1](Equal msets): Two msets $A, B \in M(X)$ are said to be equal, denoted $A = B$ if and only if for any objects $x \in X, C_A(x) = C_B(x)$. This is to say that $A = B$ if the multiplicity of every element in A is equal to its multiplicity in B and conversely.

Note: $A = B \Longrightarrow A^* = B^*$, though the converse need not hold. For example,

let $A = [a, a, b, b, c]\ and\ B = [a, a, b. b, b, c, c]\ where\ A^* = B^* = \{a, b, c\}\ but\ A \neq B$.

Definition 2.1.6[1](Submset): Let $A, B \in M(X)$. A is a *submset* of B, denoted by $A \subseteq B\ or\ B \supseteq A$, if $C_A(x) \leq C_B(x)\ for\ all\ x \in X$. Also if $A \subseteq B\ and\ A \neq B$, then A is called proper submset of B denoted by $A \subset B$. In other words $A \subset B$ if $A \subseteq B$ and there exist at least an $x \in X$ such that $C_A(x) < C_B(x)$. We assert that an mset B is called the parent mset in relation to the mset A.

Note that for any two msets $A, B \in M(X)$, $A = B$ if and only if $A \subseteq B$ and. $B \subseteq A$.

Definition. 2.1.7[1](Regular or Constant mset): An mset $A \in M(X)$ is called regular or constant if for any $x, y \in A\ such\ that\ x \neq y$ then $C_A(x) = C_A(y)$

Definition 2.1.8 [9] *($\wedge$ and $\vee$ notations)*: The notations $\wedge$ and $\vee$ denote the minimum and maximum operators respectively, for instance;

$C_A(x) \wedge C_A(y) = min\{C_A(x), C_A(y)\}\ and\ C_A(x) \vee C_A(y) = max\{C_A(x), C_A(y)\}$.

2.2 Multiset operations.

Definition 2.2.1[9] (msets union): Let $A, B \in M(X)$. The union of $A \text{ and } B$ denoted $A \cup B$ is the mset defined by $C_{A\cup B}(x) = C_A(x) \vee C_B(x)$,

Definition 2.2.2[9] (msets intersection): Let $A, B \in M(X)$.The intersection of two mset A and B denoted by $A \cap B$, is the mset for which

$C_{A\cap B}(x) = C_A(x) \wedge C_B(x) \ \ \forall x \in X.$

Definition 2.2.3[9] (mset addition): Let $A, B \in M(X)$. The direct sum or arithmetic addition of A and B denoted by $A + B$ or $A \uplus B$ is the mset defined by

$C_{A+B}(x) = C_A(x) + C_B(x) \ \ \forall x \in X.$

Note: $| A \uplus B | = | A \cup B | + | A \cap B |$.

Definition 2.2.4[9] (mset difference): Let $A, B \in M(X)$, then the difference of B from A, denoted by $A - B$ is the mset such that $C_{A-B}(x) = (C_A(x) - C_B(x)) \vee 0 \ \forall x \in X$. If $B \subseteq A$, then

$C_{A-B}(x) = C_A(x) - C_B(x).$

It is sometimes called the arithmetic difference of B from A. If $B \nsubseteq A$ this definition still holds. It follows that the deletion of an element x from an mset A give rise to a new mset $A' = A - x$ such that $C_{A'}(x) = C_A(x) - C_B(x) \vee 0$.

Definition 2.2.5[8] (mset symmetric difference): Let X be a set and $A, B \in M(X)$ Then the symmetric difference, denoted $A\Delta B$, is defined by $C_{A\Delta B}(x) = |C_A(x) - C_B(x)|$.

Note: $A\Delta B = (A - B) \cup (B - A)$.

Definition 2.2.6[8] (mset complement): Let $G = \{A_1, A_2, \dots, A_n\}$ be a family of finite msets generated from the set X. Then, the maximum mset Z is defined by $C_Z(x) = \vee_{A\in G} C_A(x)$ for all $A \in G$ and $x \in X$. The Complement of an mset A, denoted by $\bar{A}$, is defined:

$\bar{A} = Z - A \text{ such that } C_{\bar{A}}(x) = C_Z(x) - C_A(x), for\ all\ x \in X.$

Note that $A_i \subseteq Z\ for\ all\ i$.

Definition 2.2.7[8] (Multiplication by Scalar): Let $A \in M(X)$, then the scalar multiplication denoted by $b.A$ is defined by $C_{b.A}(x) = b.C_A(x),\ and\ b \in \{1,2,3, \dots\}$.

Definition 2.2.8[8] (Arithmetic Multiplication): Let $A, B \in M(X)$, then the Arithmetic Multiplication denoted by $A.B$ is defined by $C_{A.B}(x) = C_A(x).C_B(x) \ \ \forall x \in X$.

Definition 2.2.9[7] (Raising to an Arithmetic Power): Let $A \in M(X)$, then A raised to power n denoted by A^n **is** defined:

$C_{A^n}(x) = \left(C_A(x)\right)^n\ for\ n \in \{0,1,2,3, \dots\}.$

Proposition 2.2.10: Let X be a set and let $A \in M(X)$. Then

(i) $A^* = A^0$.
(ii) $A^n . A^m = A^{n+m}$, and
(iii) $(A.B)^n = A^n . B^n\ for\ any\ n, m \in \{0,1,2, \dots\}$

Proof:

(i) Let $A \in M(X)$ and let $A \neq \emptyset$. We show that $A^0 = A^*$.

Now let $x \in A^0$, then by definition 2.1.2, $C_{A^0}(x) > 0$ and by definition 2.2.9 $\left(C_A(x)\right)^0 > 0$.

Which implies that $C_A(x) > 0$. That is $x \in A^*$.

In particular, $A^0 \subseteq A^*$ (1)

Again, let $z \in A^*$, then by definition 2.1.4 $C_A(z) > 0$. But $\left(C_A(z)\right)^0 = 1 > 0$. That is $C_{A^0}(z) = 1 > 0$ which implies that $z \in A^0$

In particular, $A^* \subseteq A^0$ (2)

Hence from (1) and (2) the result $A^* = A^0$ is clear.

(ii) Let $A \in M(X)$ such that $A \neq \emptyset$. We show that $A^n . A^m = A^{n+m}$ $for\ any\ n, m \in \{0,1,2, \dots\}$

Now $C_{(A^n . A^m)}(x) = C_{A^n}(x). C_{A^m}(x)$ (by definition 2.2.8).

But $C_{A^n}(x). C_{A^m}(x) = \left(C_A(x)\right)^n . \left(C_A(x)\right)^m$ (by definition 2.2.9)

Clearly $\left(C_A(x)\right)^n . \left(C_A(x)\right)^m = \left(C_A(x)\right)^{n+m}$

and $\left(C_A(x)\right)^{n+m} = C_{A^{n+m}}(x)$

Thus, $C_{(A^n . A^m)}(x) = C_{A^{n+m}}(x)$

and $A^n . A^m = A^{n+m}$.

(iii) Let $A, B \in M(X)$ such that $A, B \neq \emptyset$ $for\ any\ n \in \{0,1,2, \dots\}$

We need show that $(A.B)^n = A^n . B^n$.

Now $C_{(A.B)^n}(x) = \left(C_{A.B}(x)\right)^n$ (by definition 2.2.9), and

$\left(C_{A.B}(x)\right)^n = \left(C_A(x). C_B(x)\right)^n$ (by definition 2.2.8)

But $\left(C_A(x). C_B(x)\right)^n = \left(C_A(x)\right)^n . \left(C_B(x)\right)^n$.

and $\left(C_A(x)\right)^n . \left(C_B(x)\right)^n = C_{A^n}(x). C_{B^n}(x)$.

In particular $C_{A^n}(x). C_{B^n}(x) = C_{A^n . B^n}(x)$ i.e $C_{(A.B)^n}(x) = C_{A^n . B^n}(x)$

Hence $(A.B)^n = A^n . B^n$.

Proposition 2.2.11: Let $A \in M(X)$ such that $A \neq \emptyset$, then $(A^n)^* = A^*$ for $n \in \{0,1,2 \dots\}$

Proof: Now let $x \in (A^n)^*$, then $C_{A^n}(x) > 0$. That is $\left(C_A(x)\right)^n > 0$.

But $\left(C_A(x)\right)^n > 0 \Rightarrow C_A(x) > 0$ (by definition 2.1.4). Thus $x \in A^*$

In particular $(A^n)^* \subseteq A^*$ (1)

Similarly let $y \in A^*$. Then $C_A(y) > 0$ (by definition 2.1.4)

Clearly $C_A(y) > 0 \Rightarrow (C_A(y))^n > 0 \; for \; n = \{0,1,2 \ldots\}$

But $\left(C_A(y)\right)^n > 0 \Rightarrow C_{A^n}(y) > 0 \Rightarrow y \in (A^n)^*$

In particular $y \in (A^n)^*$ (by definition 2.2.9)

Thus $A^* \subseteq (A^n)^*$ (2)

Now from (1) and (2) above, it is clear that $A^* = (A^n)^*$

2.3 Mset functions.

Definition 2.3.1[17]: Let X be a non empty set and let $A, B \in M(X)$. We defined the mset function $f: A \to B$ as just the function $f: A^* \to B^*$ such that for any $x \in X$, $C_{f(A)}\big(f(x)\big) = C_A(x)$. The image of an mset $A \in M(X)$ under an mset function f denoted by $f(A)$ is given by

$$f(A) = \left\{\frac{m_i}{f(x_i)} : \; x \in A, \; m_i = C_{f(A)}\big(f(x_i)\big) = C_A(x_i)\right\}.$$

Definition 2.3.2[17](Injective mset function): Let $A, B \in M(X)$. The mset function $f: A \to B$ is said to be injective if

(i) $f: A^* \to B^*$ is injective and

(ii) $\forall x (x \in A^* \Rightarrow C_A(x) \leq C_B\big(f(x)\big)$.

Definition 2.3.3[17](Surjective mset function): Let $A, B \in M(X)$. The mset function $f: A \to B$ is said to be surjective if

(i) $f: A^* \to B^*$ is surjective and

(ii) $\forall x (x \in A^* \Rightarrow C_A(x) \geq C_B\big(f(x)\big)$.

Definition 2.3.4[17](Bijective mset function): Let $A, B \in M(X)$. The mset function $f: A \to B$ is said to be a bijective if

(i) $f: A^* \to B^*$ is bijective and

(ii) $\forall x (x \in A^* \Rightarrow C_A(x) = C_B\big(f(x)\big)$.

Note that a bijective mset function is both injective and surjective

Definition 2.3.5[17](Inverse mset function): Let $A, B \in M(X)$. The inverse of the mset function $f: A \to B$ denoted f^{-1} is just the mset function $f^{-1}: B \to A$.

Definition 2.3.6[17](Composition of mset function): Let $A, B \; and \; C \in M(X)$. We defined the composition of mset function of $f: A \to B$ and $g: B \to C$ denoted as $gof: A \to C$ as just the composition $gof: A^* \to C^*$, such that $C_{gof(A)}\big(gof(x)\big) = C_A(x)$.

Theorem 2.3.7[17]: Let $A, B \in M(X)$. If the mset function $f: A \to B$ is bijective, then $f^{-1}: B \to A$ is bijective.

Theorem 2.3.8[17]: Let $A, B, C \in M(X)$, if $f: A \to B$ and $g: B \to C$ are injective mset functions, then $gof: A \to C$ is injective.

Theorem 2.3.9[17]: 16: Let $A, B, C \in M(X)$, if $f: A \to B$ and $g: B \to C$ are surjective mset functions, then $gof: A \to C$ is surjective.

Theorem 2.3.10[17]: 16: Let $A, B, C \in M(X)$, if $f: A \to B$ and $g: B \to C$ are bijective mset functions, then $gof: A \to C$ is bijective.

Proposition 2.3.11: Let $A, B \in M(X)$ where A is regular such that $f: A \to B$ is an mset function. Then $f(A)$ is regular.

Proof:

Let $A \in M(X)$ be regular.

We show that for any $y_1, y_2 \in f(A)$ such that $y_1 \neq y_2$, we have $c_{(f(A))}(y_1) = c_{(f(A))}(y_2)$.

Now $y_1, y_2 \in f(A)$ implies the existence of $x_1, x_2 \in A$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

But $C_A(x_1) = C_A(x_2)$ (by definition 2.1.7)

Thus, $c_{(f(A))}(y_1) = c_{(f(A))}(y_2)$ (by definition 2.3.1)

In particular $f(A)$ is regular.

3. mgroupoids.

Definition 3.1.1: Let X be a groupoid, and $A \in M(X)$. A is said to be an mgroupoid if the following condition is satisfied.

$C_A(xy) \geq C_A(x) \wedge C_A(y), \forall\, x, y \in X.$

Example 3.1.2: Given a groupoid $X = Z_3 = \{0,1,2\}$ under addition let $A = \{0,1,2\}_{4,3,1}$. Then

$$C_A(0+1) = C_A(1) = 3 \geq min\{C_A(0), C_A(1)\} = min\{4,3\} = 3.$$

$$C_A(0+2) = C_A(2) = 1 \geq min\{C_A(0), C_A(2)\} = min\{4,1\} = 1.$$

$$C_A(1+2) = C_A(0) = 4 \geq min\{C_A(1), C_A(2)\} = min\{3,1\} = 1.$$

Thus A is a mgroupoid.

We denote the class of all mgroupoid over X by $MGP(X)$.

Definition 3.1.3[6](Composition of mgroupoids): Let $A, B \in MGP(X)$, then the composition of A and B denoted $A \circ B$ is defined:

$C_{A\circ B}(x) = \vee\{C_A(y) \wedge C_B(z): y, z \in X \ni yz = x\}$

Proposition 3.1.4: Let $A \in MGP(X)$. Then A^* is a sub groupoid of X.

Proof: Supposing $A \in MGP(X)$. Let $x, y \in A^*$, then $C_A(x), C_A(x) > 0 (by\ definition\ 2.1.4)$.

In particular $C_A(x) \wedge C_A(y) > 0$. But $A \in MGP(X)$ implies

$C_A(xy) \geq C_A(x) \wedge C_A(y) > 0\ \forall\, x, y \in X$

Thus $C_A(xy) > 0$ i.e $xy \in A^*$.

In particular A^* is a sub groupoid of X (since $A^* \subseteq X$)

Proposition 3.1.5: For any $A \in MGP(X)$, $A^* \in MGP(X)$.

Proof: Let $x, y \in A^*$. We want to show that

$C_{A^*}(xy) \geq C_{A^*}(x) \wedge C_{A^*}(y), \forall\, x, y \in X \qquad (*)$

From the following possibilities:

(i) $x, y \in A^* \Rightarrow xy \in A^*$ (from proposition 3.1.4)

(ii) $x \in A^*$ and $y \notin A^* \Rightarrow xy \in A^*$ or $xy \notin A^*$
(iii) $x \notin A^*$ and $y \in A^* \Rightarrow xy \in A^*$ or $xy \notin A^*$
(iv) $x \notin A^*$ and $y \notin A^* \Rightarrow xy \in A^*$ or $xy \notin A^*$

the inequality $(*)$ is valid, from (i) - (iv) above.

Thus $C_{A^*}(xy) \geq C_{A^*}(x) \wedge C_{A^*}(y), \forall\, x, y \in X$

In particular, $A^* \in MGP(X)$.

Proposition 3.1.6: Let $A \in MGP(X)$. Then $A^0 \in MGP(X)$.

Proof: Supposing $A \in MGP(X)$. Then $A^0 = A^*$ (from proposition 2.2.10)

Thus from proposition 3.1.5, the result follows.

3.2 Closure of mset operations on MGP(X).

Proposition 3.2.1: Let X be a groupoid and let $A, B \in MGP(X)$, Then $A \cap B \in MGP(X)$.

Proof: Given that $A, B \in MGP(X)$, then we have $C_A(xy) \geq C_A(x) \wedge C_A(y)\ \forall x, y \in X$ and $C_B(xy) \geq C_B(x) \wedge C_B(y)\ \forall x, y \in X$.

Now, $C_{A\cap B}(xy) = \{[C_A(xy) \wedge C_B(xy)]\}$ (by definition)

$\geq [(C_A(x) \wedge C_A(y)) \wedge (C_B(x) \wedge C_B(y))]$ (by definition)

$= [C_A(x) \wedge C_B(x)] \wedge [C_A(y) \wedge C_B(y)] (by\ hypothesis)$

$= C_{A\cap B}(x) \wedge C_{A\cap B}(y)$

Thus $C_{A\cap B}(xy) \geq C_{A\cap B}(x) \wedge C_{A\cap B}(y)$.

In particular, $A \cap B \in MGP(X)$.

Generally, If $\{A_i : i \in I\}$ be a family of mgroupoids over a groupoid X, then their intersection $\cap_{i\in I} A_i$ is a mgroupoid over X, i.e $\bigcap_{i\in I} A_i \in MGP(X)$

Remark 1: Let X be a groupoid and let $A, B \in MGP(X)$, Then $A \cup B$ need not be a mgroupoid.

For example: Let $X = \{e, a, b, c\}$ be a groupoid with $a^2 = b^2 = c^2 = e^2 = e\ and$

$ab = ba = c, ac = ca = b, bc = cb = a$ and e is the identity element. If $A = \{e, a, b, c\}_{3,2,1,2}$, and $B = \{e, a, b, c\}_{3,1,1,2}$. Clearly X is a groupoid, and $A \cup B = \{e, a, b, c\}_{3,2,1,2}$.

Now

$C_{A\cup B}(ac) = C_{A\cup B}(b) = 1 \ngeq [C_{A\cup B}(a) \wedge C_{A\cup B}(c)] = min[C_{A\cup B}(a), C_{A\cup B}(c)] = min[2,2] = 2.$

Hence $A \cup B \notin MGP(X)$.

Generally, let $\{A_i : i \in I\}$ be a family of mgroupoid over a groupoid X, then their union $\cup_{i\in I} A_i$ need not be a mgroupoid over X, i.e $\bigcup_{i=I} A_i \notin MGP(X)$.

Remark 2: Let $A, B \in MGP(X)$, then $A + B$ need not be a mgroupoid.

For example Let X be as in the example of remark 1. If $A = \{e, a, b, c\}_{4,3,3,1}$ and $B = \{e, a, b, c\}_{3,2,1,1}$. Then $A + B = \{e, a, b, c\}_{7,5,4,2}$.

Now

$$C_{A+B}(ab) = C_{A+B}(c) = 2 \ngeq C_{A+B}(a) \wedge C_{A+B}(b) = \min\{C_{A+B}(a), C_{A+B}(b)\} = \min\{5.4\} = 4$$

Showing that $A + B \notin MGP(X)$.

Remark 3:: Let $A, B \in MGP(X)$, then $A - B$ need not be a mgroupoid..

For example Let X be as in the example of remark 1. If $A = \{e, a, b, c\}_{5,3,2,2}$ and $B = \{e, a, b, c\}_{3,2,2,1}$. Then $A - B = \{e, a, b, c\}_{2,1,0,1}$.

Now
$C_{A-B}(ac) = C_{A-B}(b) = 0 \ngeq C_{A-B}(a) \wedge C_{A-B}(c) = \min\{C_{A-B}(a), C_{A-B}(c)\} = \min\{2.1\} = 1$

Showing that $A - B \notin MGP(X)$.

Remark 4: Let $A, B \in MGP(X)$, then $A\Delta B$ need not be a mgroupoid.

For example Let X be as in the example of remark 1. If $A = \{e, a, b, c\}_{5,3,2,2}$ and $B = \{e, a, b, c\}_{3,2,2,1}$. Then $A\Delta B = \{e, a, b, c\}_{2,1,0,1}$.

Now
$|C_{A\Delta B}(ac)| = |C_{A\Delta B}(b)| = |0| = 0 \ngeq |C_{A\Delta B}(a) \wedge C_{A\Delta B}(c)| = \min\{C_{A\Delta B}(a), C_{A\Delta B}(c)\} = \min\{1.1\} = |1| = 1$

Thus $A\Delta B \notin MGP(X)$.

Remark 5: Let $A \in MGP(X)$, then Â need not be a mgroupoid.

Given that $\bar{A} = Z - A$, (by definition 2.2.6) we show that $\bar{A} \notin MGP(X)$ by the given example:

Given the groupoid $X = Z_4 = \{0,1,2,3\}$ under additive operation.

Let $Z = \{0,1,2,3\}_{5,4,3,1}$ and $A = \{0,1,2,3\}_{3,2,2,1}$. Thus $\bar{A} = \{0,1,2,3\}_{2,2,1,0}$

Now
$C_{\bar{A}}(1 + 2) = C_{\bar{A}}(3) = 0 \ngeq C_{\bar{A}}(1) \wedge C_{\bar{A}}(2) = \min\{C_{\bar{A}}(1), C_{\bar{A}}(2)\} = \min\{2.1\} = 1$

That is $\bar{A} \notin MGP(X)$.

Proposition 3.2.2: Let $A \in MGP(X)$ then the scalar multiplication $b.A \in MGP(X)$, $b \in \mathbb{N}$ (the set of natural numbers).

Proof: Let $x, y \in X$ and $b \in \mathbb{N}$. Let $A \in MGP(X)$. We want to show that $b.A \in MGP(X)$.

$C_{b.A}(xy) = b.C_A(xy)(by\ definition\ 2.2.7)$.

$\geq b.[C_A(x) \wedge C_A(y)]\ (by\ hypothesis)$

$= C_{b.A}(x) \wedge C_{b.A}(y)$

Thus $C_{b.A}(xy) \geq C_{b.A}(x) \wedge C_{b.A}\ (y)$

Hence $b.A \in MGP(X)$.

Proposition 3.2.3: Let $A, B \in MGP(X)$ then the Arithmetic Multiplication $A.B \in MGP(X)$

Proof: Since $A.B \in MGP(X)$ then $\forall\ x, y \in X$.

$C_{A.B}(xy) = C_A(xy).C_B(xy)(by\ definition\ 2.2.8)$

$\geq [C_A(x) \wedge C_A(y)].[C_B(x) \wedge C_B(y)]\ (by\ hypothesis)$

$= C_A(x).C_B(x) \wedge C_A(y).C_B(y)$

$= [C_{A.B}(x) \wedge C_{A.B}(y)]$

Showing $A.B \in MGP(X)$.

Proposition 3.2.4: Let X be a groupoid and let $A \in MGP(X)$, then $A^n \in MGP(X)$ For any $n \in \mathbb{N}$.

Proof: Let $x, y \in X$ and let $A \in MGP(X)$. We want to show that $A^n \in MGP(X)$.

Since $C_{A^n}(xy) = (C_A(xy))^n \geq [C_A(x) \wedge C_A(y)]^n$

$= [C_A(x)]^n \wedge [C_A(y)]^n$

$= C_{A^n}(x) \wedge C_{A^n}(y)$

Thus $C_{A^n}(xy) \geq C_{A^n}(x) \wedge C_{A^n}(y)$.

Hence $A^n \in MGP(X)$.

Proposition 3.2.5: Let $A, B \in MGP(X)$, then $AoB \in MGP(X)$

Proof: Let $x, y \in X$. Let $A, B \in MGP(X)$. We show that $AoB \in MGP(X)$.

Now $C_{AoB}(xy) = \vee[C_A(ab) \wedge C_B(cd); a, b, c, d \in X, (ac)(bd) = xy]$

$\geq \vee[[C_A(a) \wedge C_A(b)] \wedge [C_B(c) \wedge C_B(d)]; a, b, c, d \in X, (ac)(bd) = xy]$

$= \vee[[C_A(a) \wedge C_B(c)]; a, c \in X, (ac) = x] \wedge [C_A(b) \wedge C_B(d)]; b, d \in X, (bd) = y]]$

$= [\vee[C_A(a) \wedge C_B(c)]; a, c \in X, (ac) = x] \wedge [\vee[C_A(b) \wedge C_B(d)]; b, d \in X, (bd) = y]]]$

$= C_{AoB}(x) \wedge C_{AoB}(y)$

Thus $C_{AoB}(xy) \geq C_{AoB}(x) \wedge C_{AoB}(y)$.

Hence $AoB \in MGP(X)$.

Proposition 3.2.6: Let $x, y \in X$. Let $A, B \in MGP(X)$. Then $(A.B)^n = A^n.B^n \in MGP(X), for\ n \in \mathbb{N}$.

Proof: The result is clear from propositions 2.2.10 and 3.2.4

Proposition 3.2.7: Let $x, y \in X$. Let $A \in MGP(X)$. Then $A^n.A^m = A^{n+m} \in MGP(X), for\ n, m \in \mathbb{N}$.

Proof: Since A^n, $A^m \in MGP(X)$ (proposition 2.2.10)

Then $A^n.A^m = A^{n+m} \in MGP(X)$ (by propositions 3.2.4 and 3.2.6)

Definition 3.2.8: Let $A \in MGP(X)$ and let B be a submset of A. Then B can be said to be a sub mgroupoid of A, if $B \in MGP(X)$.

Proposition 3.2.9: Let $A \in MGP(X)$. Then $C_A(x^n) \geq C_A(x), \forall\ x \in X$

Proof: Let $x, y \in X$. Then

$C_A(x^n) = C_A(x^{n-1}x) \geq C_A(x^{n-1}) \wedge C_A(x)$

$= C_A(x^{n-2}x) \wedge C_A(x) \wedge C_A(x) \geq C_A(x^{n-2}) \wedge C_A(x) \wedge C_A(x)$

$\geq C_A(x) \wedge C_A(x) \wedge C_A(x) \wedge C_A(x) \wedge \ldots \wedge C_A(x) = C_A(x), \forall\ x \in X.$

Thus $C_A(x^n) \geq C_A(x), \forall\ x \in X$.

3.3 Cancellability.

Definition 3.3.1: Let $A \in MGP(X)$ an element $a \in A$ is said to be cancellable if

$C_A(ax) = C_A(ay),\ and\ C_A(xa) = C_A(ya)$, implies $C_A(x) = C_A(y)$.

Definition 3.3.2: Let $A \in MGP(X)$. Then A is said to be cancellable if a is cancellable for all $a \in A$..

Proposition 3.3.3: Let $A \in MGP(X)$. If A is regular, then A is cancellable.

Proof: Let $A \in MGP(X)$ be regular and $a, x, y \in A$ such that

$C_A(ax) = C_A(ay)$ and $C_A(xa) = C_A(ya)$ then we show that $C_A(x) = C_A(y)$.

But $ax, ay, xa,$ and $ya \in A^*$, since A^* is a subgroupoid (from proposition 3.1.4).

thus $C_A(ax) = C_A(x),\ C_A(xa) = C_A(x)$ and $C_A(ay) = C_A(y),\ C_A(ya) = C_A(y)$ (since A is regular),

That is $C_A(x) = C_A(ax) = C_A(xa) = C_A(ay) = C_A(ya) = C_A(y)$

Thus $C_A(x) = C_A(y)$. Hence A is cancellable (following the arbitrary choice of $\in A$).

Proposition 3.3.4: Let $A \in MGP(X)$, then A is regular if and only if A is cancellable.

Proof: Supposing $A \in MGP(X)$ is regular, then A is cancellable (From proposition 3.3.3).

Conversely, assuming that A is cancellable but not regular. Therefore there exist $x, y \in A$, such that

$C_A(x) \neq C_A(y)$ (1)

However, let $C_A(ax) = C_A(ay)$ and $C_A(ax) = C_A(ay)$ for $a \in A$.

Then $C_A(x) = C_A(y)$. (Since A is cancellable by hypothesis) (2)

From (1) and (2), we have a contradiction and the result follows.

Definition 3.3.5: Let $A \in MGP(X)$, then A is said to be a commutative mgroupoid if

$C_A(xy) = C_A(yx)\ \forall\, x, y \in X.$

Commutative mgroupoid can also be called Abelian mgroupoid.

Example: Let $X = \{e, a, b, c\}$, with $a^2 = b^2 = c^2 = e^2 = e\ and\ ab = ba = c,$

$ac = ca = b, bc = cb = a$. Where e is the identity element. If $A = \{e, a, b, c\}_{3,2,3,2}$ is an mset over X.

Clearly A is a commutative mgroupoid.

Proposition 3.3.6: Let $A \in MGP(X)$. If A is regular, then A is commutative.

Proof: Let A be regular. We show that A is commutative

Now for all $x, y \in X$ such that $x \neq y$ we have $C_A(x) = C_A(y)$(by hypothesis and definition)

Since X is a groupoid, then $xy, yx \in X$, and $C_A(xy) = C_A(yx)$ (since A is regular)

In particular, A is commutative (by definition)

Proposition 3.3.7: Let $A \in MGP(X)$. Then A is commutative if and only if A is regular.

Proof: Supposing A is commutative and not regular, that is

$C_A(xy) = C_A(yx)\ \forall\, x, y \in X$ (by definition and hypothesis) (1)

But $xy, yx \in X$ (by hypothesis and definition)

Let $xy = Z_1$ *and* $yx = Z_2$. If $Z_1 \neq Z_2$, then $C_A(Z_1) = C_A(Z_2)$. (from (1) above)

A contradiction, since A is not regular by hypothesis.

The result follows.

Conversely, given that A is regular, then A is commutative (by proposition 3.3.7).

Proposition 3.3.8: Let $A \in MGP(X)$. If A is a commutative mgroupoid, then A^* is a commutative sub mgroupoid.

Proof: Let A be commutative.

i.e $C_A(xy) = C_A(yx)\ \forall\ x, y \in X$ (by definition). (1)

We show that A^* is commutative.

i.e $C_{A^*}(xy) = C_{A^*}(yx)\ \forall\ x, y \in X$

Supposing $C_{A^*}(xy) \neq C_{A^*}(yx)$, then either

$C_{A^*}(xy) > C_{A^*}(yx)$ or (2)

$C_{A^*}(xy) < C_{A^*}(yx)$ (3)

But $C_{A^*}(xy), C_{A^*}(yx) \in \{0,1\}$

From (ii) $C_{A^*}(yx) = 0$ *and* $C_{A^*}(xy) = 1$

$C_A(yx) = 0$ *and* $C_A(xy) > 0$ (4)

From (1) and (4). We have a contradiction

Similarly, for (3) above

Hence, the result follows.

3.4 Homomorphism of mgroupoids.

Definition 3.4.1: Let X be groupoid. If $A, B \in M(X)$, the mset function $f: A \to B$ is said to be a homomorphism of mgroupoids if and only if

(i) $f: A^* \to B^*$ is a homomorphism of groupoids and
(ii) $A, B \in MGP(X)$

Example 3.4.2: Using the groupoid $X = Z_4 = \{0,1,2,3\}$ under the multiplicative operation and the mgroupoids $A = \{0,1,2\}_{4,2,2}$ and $B = \{0,1,2,3\}_{3,3,2,2}$ over X. We defined our function $f: A^* \to B^*$ as $f(x) = x^3$. Clearly, $f(xy) = (xy)^3 = x^3y^3 = f(x)f(y)$.

Clearly, A^*, B^* are groupoids and $f: A^* \to B^*$ is a homomorphism of groupoids.

Now

$C_A(0.1) = C_A(0) = 4 \geq C_A(0) \wedge C_A(1) = \min\{4,2\} = 2$

$C_A(0.2) = C_A(0) = 4 \geq C_A(0) \wedge C_A(2) = \min\{4,2\} = 2$

$C_A(1.2) = C_A(2) = 2 \geq C_A(1) \wedge C_A(2) = \min\{2,2\} = 2$

And

$C_B(0.1) = C_B(0) = 3 \geq C_B(0) \wedge C_B(1) = \min\{3,3\} = 3$

$C_B(0.2) = C_B(0) = 3 \geq C_B(0) \wedge C_B(2) = \min\{3,2\} = 2.$

$C_B(0.3) = C_B(0) = 3 \geq C_B(0) \wedge C_B(3) = \min\{3,2\} = 2.$

$C_B(1.2) = C_B(2) = 2 \geq C_B(1) \wedge C_B(2) = \min\{3,2\} = 2.$

$C_B(1.3) = C_B(3) = 2 \geq C_B(1) \wedge C_B(3) = \min\{3,2\} = 2.$

$C_B(2.3) = C_B(2) = 2 \geq C_B(2) \wedge C_B(3) = \min\{2,2\} = 2.$

In particular, $A, B \in MGP(Z_4)$.

Thus f is a homomorphism of mgroupoids.

We denote the class of homomorphisms of mgroupoids over $MGP(X)$ by $HMGP(X)$

Proposition 3.4.3: Let X be a groupoid and $A, B \in M(X)$. If $f: A \to B$ is a homomorphism of mgroupoids then $f(A) \in MGP(X)$.

Proof: Let $f: A \to B$ be a homomorphism of mgroupoids. Then $A, B \in MGP(X)$ (by definition 3.4.1). Let $x, y \in X$, then

$C_{f(A)}\big(f(xy)\big) = C_A(xy) \geq C_A(x) \wedge C_A(y)$(by definition 2.3.1)

But $C_A(x) = C_{f(A)}\big(f(x)\big)$ and $C_A(y) = C_{f(A)}\big(f(y)\big)$ (by definition 2.3.1)

Thus $C_{f(A)}\big(f(xy)\big) \geq C_{f(A)}\big(f(x)\big) \wedge C_{f(A)}\big(f(y)\big)$.

i.e $C_{f(A)}[f(x)f(y)] \geq C_{f(A)}\big(f(x)\big) \wedge C_{f(A)}\big(f(y)\big)$.

Hence $f(A) \in MGP(X)$.

Proposition 3.4.4: Let $A, B \in MGP(X)$, If $f \in HMGP(X)$ such that $f: A \to B$ is bijective, then $f^{-1} \in HMGP(X)$ such that $f^{-1}: B \to A$ is bijective and $f^{-1}(B) \in MGP(X)$.

Proof: Clearly, $f^{-1}: B \to A$ is bijective (Theorem 2.3.7) (1)

Then we show that $f^{-1}: B \to A$ is a homomorphism of mgroupoids.

Now for any $y_1, y_2 \in B^*$, there exist $x_1, x_2 \in A^*$ such that $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$

But $f(x_1x_2) = f(x_1)f(x_2) = y_1y_2$, (since f is a homomorphism)

Then $f^{-1}(y_1y_2) = f^{-1}(y_1)f^{-1}(y_2) = x_1x_2$

Therefore $f^{-1}: B^* \to A^*$ is a homomorphism of groupoids. (2)

Now $C_{f^{-1}(B)}\big(f^{-1}(y_1y_2)\big) = C_B(y_1y_2)$ (by definition 2.3.1)

But $C_B(y_1y_2) \geq C_B(y_1) \wedge C_B(y_2)$, $(Since\ B \in MGP(X))$

But $C_B(y_1) = C_{f^{-1}(B)}\big(f^{-1}(y_1)\big)$ and $C_B(y_2) = C_{f^{-1}(B)}\big(f^{-1}(y_2)\big)$(by definition 2.3.1)

Thus $C_{f^{-1}(B)}\big(f^{-1}(y_1y_2)\big) \geq C_{f^{-1}(B)}\big(f^{-1}(y_1)\big) \wedge C_{f^{-1}(B)}\big(f^{-1}(y_2)\big)$

i.e $C_{f^{-1}(B)}\big(f^{-1}(y_1)f^{-1}(y_2)\big) \geq C_{f^{-1}(B)}\big(f^{-1}(y_1)\big) \wedge C_{f^{-1}(B)}\big(f^{-1}(y_2)\big)$ (3)

Hence $f^{-1}(B) \in MGP(X)$(from proposition 3.4.3) and $f^{-1} \in HMGP(X)$ following (1), (2) and (3) above

Corollary 3.4.5: Let X groupoid and let $A_i, B_i \in MGP(X)$. If $f \in HMGP(X)$ such that

$f{:}\cap_{i\in I} A_i \to \cap_{i\in I} B_i$, then $f(\cap_{i\in I} A_i) \in MGP(X)$, and If f is bijective, then $f^{-1}(\cap_{i\in I} B_i) \in MGP(X)$.

Proof: The results follows from proposition 3.4.3 and proposition 3.4.4

Proposition 3.4.6: Let $A, B \in MGP(X)$, If $f \in HMGP(X)$ such that $f: A \to B$, then $f: A^n \to B^n$ is a homomorphism of mgroupoids and $f(A^n) \in MGP(X)$, for

$n \in \{0,1,2, \dots\}$.

Proof: Let $f: A \to B$ be a homomorphism of mgroupoids, then we show that $f: A^n \to B^n$ is equally a homomorphism of mgroupoids.

Note that $A^n, B^n \in MGP(X)$ given that $A, B \in MGP(X)$ (by proposition 3.2.4) (1)

Now given that $f: A \to B$ is a homomorphism of mgroupoids, then $f: A^* \to B^*$ is a homomorphism of groupoids (by definition 3.4.1) (2)

Note that $(A^n)^* = A^*$ and $(B^n)^* = B^*$ (see proposition 2.2.11) (3)

Thus $f: (A^n)^* = A^* \to (B^n)^* = B^*$ is a homomorphism of groupoids (from (3)) (4)

It is clear from (1) and (4), that $f: A^n \to B^n$ is a homomorphism of mgroupoids.

And $f(A^n) \in MGP(X)$(from proposition 3.4.3)

Proposition 3.4.7: Let $A, B, C \in M(X)$, If $f: A \to B$ and $g: B \to C$ are homomorphism of mgroupoids then

(i) $g \circ f: A \to C$ is a homomorphism of mgroupoids.
(ii) $g \circ f(A) \in MGP(X)$.

Proof: Let $f: A \to B$ and $g: B \to C$ be homomorphism of mgroupoids.

Clearly $A, B, C \in MGP(X)$

Since $f: A \to B$ and $g: B \to C$ are homomorphism of mgroupoids, then $f: A^* \to B^*$ and $g: B^* \to C^*$ are homomorphism of groupoids. (ii)

From (i) and (ii) above, it is clear that $g \circ f: A \to C$ is a homomorphism of mgroupoids, and

$g \circ f(A) \in MGP(X)$. The result.

Proposition 3.4.8: Let $A, B \in MGP(X)$ and $f: A \to B$ a homomorphism of mgroupoids. If $a \in A$ is cancellable, then $f(a)$ is cancellable.

Proof: Let $A, B \in MGP(X)$ and let $a \in A$ be cancellable. We show that, $f(a)$ is cancellable.

Given that $a \in A$ is cancellable, then for any $x, y \in A$ such that $C_A(ax) = C_A(ay), C_A(xa) = C_A(ya)$ we have

$C_A(x) = C_A(y)$.(by definition) (1)

But, $C_{f(A)}\big(f(ax)\big) = C_A(ax)$, $C_{f(A)}\big(f(ay)\big) = C_A(ay)$, $C_{f(A)}\big(f(x)\big) = C_A(x)$

and $C_{f(A)}\big(f(y)\big) = C_A(y)$ (By definition 3.4.1) (2)

Also, supposing $C_{f(A)}\big(f(a)f(x)\big) = C_{f(A)}\big(f(a)f(y)\big)$ and $C_{f(A)}\big(f(x)f(a)\big) = C_{f(A)}\big(f(y)f(a)\big)$ i.e $C_{f(A)}\big(f(ax)\big) = C_{f(A)}\big(f(ay)\big)\ and\ C_{f(A)}\big(f(xa)\big) = C_{f(A)}\big(f(ya)\big)$ (f is a homomorphism) i.e $C_A(ax) = C_A(ay)$ *and* $C_A(xa) = C_A(ya)$ (by definition)

But a is cancellable, therefore $C_A(x) = C_A(y)$(by definition) (3)

But $C_A(x) = C_{f(A)}\big(f(x)\big)$ and $C_A(y) = C_{f(A)}\big(f(y)\big)$ (by definition)

Therefore $C_{f(A)}\big(f(x)\big) = C_{f(A)}\big(f(y)\big)$ (from 3)

Hence $f(a)$ is cancellable.

Proposition 3.4.9: Let $A, B \in MGP(X)$. If A is cancellable such that $f: A \to B$ a homomorphism of mgroupoids, then $f(A)$ is cancellable.

Proof: Supposing $f(A)$ is not cancellable when A is cancellable then there exist $y_1 \in f(A)$ that is not cancellable.

Since $y_1 \in f(A)$, there exist $a \in A$ such that $f(a) = y_1$. But $a \in A$ is cancellable (since A cancellable). Thus $f(a) = y_1$ is cancellable (from proposition 3.3.4).

But $f(a) = y_1$ is not cancellable.(from the hypothesis), a contradiction.

Thus $f(A)$ is cancellable.

Proposition 3.4.10: Let $A, B \in MGP(X)$ such that A is commutative. If $f: A \to B$ is a homomorphism, then $f(A)$ is a commutative mgroupoid.

Proof: Let $A, B \in MGP(X)$ such that A is commutative, then

$C_A(xy) = C_A(yx) \;\; \forall\, x, y \in X$ (by definition)

We show that $f(A)$ is commutative.

Now, $C_{f(A)}(f(x)) = C_A(x)$ *and* $C_{f(A)}(f(y)) = C_A(y), \forall\, x, y \in X$(by definition)

Also $C_{f(A)}(f(x)f(y)) = C_{f(A)}(f(xy))$ (f is a homomorphism) (1)

And $C_{f(A)}(f(xy)) = C_A(xy)$(by definition). (2)

But $C_A(xy) = C_A(yx)$ (by hypothesis) (3)

From (3), (2) and (1) above, we hhave

$C_{f(A)}(f(x)f(y)) = C_{f(A)}(f(y)f(x))$.

Hence $f(A)$ is commutative.

4.0 Conclusion.

We have introduced and studied the concepts of mgroupoid. In the study, we have established the closure of some multiset operations over the class of finite mgroupoids. We have established that the root set of an mgroupoid is a subgroupoid and sub mgroupoid. Cancellation law was introduced and studied and we established that a homomorphic image of a cancellable element or a cancellable mgroupoid are cancellable and also shown that an mgroupoid is regular if and only if it is cancellable. We have also introduced and study the homomorphism of mgroupoids where it is established that the image of a mgroupoid under a homomorphism is closed in the class of mgroupoids. The composition of homomorphisms of mgroupoids is closed in the class of homomorphisms of mgroupoids. We also established the commutativity of the homomorphic image of a commutative mgroupoids.

References.

[1] Blizard, W..(1989); Multiset Theory, Notre Dame, J. Formal Logic 30, 36-66.

[2] Cerf V., Fenandez E., Gostelow K. and Volausky S., Formal control and low properties of a model of computation, Report ENG 7178, Computer Science Department, University of California, Los Angeles, CA, December, pp. 81, (1971).

[3] K.P. Girish and Sunil Jacob John, (2012); On Multiset Topologies; Theory and Application of Mathematical and Computer Sciences(2)(2012),37-52

[4] Knuth, D.E. (1981); The art of computer programming, Vol. 2 (semi numerical Algorithms), Second Edition, Addison-Wesley, Reading, Massachusetts.

[5] M. Raafat and A.M.Abd El-Latif,2010; On Multi Weak Structures; International Journal of Advances in Mathematics; Volume 2017,Number 6, 42-50.

[6] Nazmul, S.K., Majumbar, P. and Samantha, S.K. (2013) – on multiset and multigroups, Annals of Fuzzy Mathematics and Informatics, 6(3), 643-656.

[7] Parker – Rhodes, A.F. (1981); The Theory of Indistinguishables, D. Reidel, Dordrecht.

[8] Petrovsky, A. B, (1997); Structuring Techniques in Multiset Spaces. Springer. 174-184

[9] Wildberger, N.J. (2003); A new look at multiset, School of Mathematics, UNSW, Sydney.

[10] Shravan K. and Tripathy B.C., (2018); Generalised closed Sets in multiset topological spaces, Proyecciones Jour. Math. Vol. 37, 2.pp. 223-237.

[11] Singh, D., Ibrahim, A,M., Yohanna, T. and Singh, J.N. (2011) – Complemention in multiset Theory, International Mathematica Forum, 6(38), 1877.

[12] Tella, Y. (2013) – A Study of group theory in the context of Multiset Theory. International Journal of Science and Technology, 2(8), 609-615.

[13] Tella, Y. and Daniel, S. (2013); Symmetric groups under multiset perspective, Journal of Mathematics, 7(5), 47-52.

[14] Yager, R. R, (1986); On the theory of bags. International Journal of Genaral System, 13; 23-37

[15] Tero Harja, (1996); Lecture note on Semigroups, Department of Mathematics, University of Turku, FIN-20014 Turku, Finland.

[16] Benjamin Baumslag and Bruce Chanler, (1968); Theory and Problems of group theory, Schaums Outline Seriries, McGRAW-HILL BOOK COMPANY, New York, St. Louis Francisco, Toronto, Sydney.

[17] Gambo Jeremiah Gyam and Tella Yohanna, (2022); On Multiset Functions, International Journals of Sciences and Engineering Research(IJSER). Volume 13, Issue (6).