*i*Journals: International Journal of Software & Hardware Research in Engineering (IJSHRE)
ISSN-2347-4890
Volume 11 Issue 2 February 2023

New Exact Solutions Of Nonlinear Partial Differential Equations Double Sinh-Gordon Equation , By Lie Group Similarity Transformation

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DOI: 10.26821/IJSHRE.11.2.2023.10104

ABSTRACT

Using Lie Group Similarity Transformation, the nonlinear partial differential equation Double sinh-Gordon equation, $u_{tt}-u_{xx} = 8C \sinh(u) – 32C \sinh(u/2)$ is reduced to second order nonlinear ordinary differential equation and exactly solved. Each and every solution is combination of a pair of exact solutions, hyperbolically rotational invariant and other translational invariant, both exist independently and they form a new class of solutions of Klein-Gordon family of nonlinear partial differential equations. Solutions are extended to (3+1) dimensions.

Keywords:

Similarity Transformation, Nonlinear Partial Differential Equation, sinh-Gordon, hyperbolically, Rotational, Translational, Klein-Gordon, Potential, Solitary, Symmetry, Coefficient, Constrained, Generator.

01.INTRODUCTION

P.M.Morse [01] introduced a potential

$$V(u) = \frac{a}{b}\exp(-bu) - \frac{2a}{b}\exp(-\frac{b}{2}u) , \qquad (1.01)$$

, is generally known as Morse potential. Its generalized form is

$$V(u) = \frac{a}{b}\exp(-bu) + \frac{2c}{b}\exp(-bu/2) + \frac{d}{b}\exp(bu) + \frac{2e}{b}\exp(bu/2) \qquad (1.02)$$

For which the equation of motion is called Double sinh-Gordon (DShG) equation ,first introduced by S.N.Behra and A.Khare [02] and found a kink type exact solutions . Solitary wave type exact solutions of single and multi solutions reported this author [03] . both above exact solutions are of travelling wave type as function of $(kx- vt)$ with translational invariant symmetry alone..M.Minami [04] also studied above DShG equation.

In this study reporting new exact solutions of DShG equation

$$u_{tt} - u_{xx} = 8C \sinh(u) – 32C \sinh(u/2) , \qquad (1.03)$$

where C is a nonzero parameter and sing Lie Group Similarity Transformation [05] method, by reducing to a second order ordinary differential (DE) equation and solved exactly. These new solution is a pair of two solutions , one is hyperbolically rotationally invariant and other is both rotationally invariant as well as translational invariant. As such these solutions form a new class of exact solutions of Klein-Gordon family of equations. Solutions are of complex valued..

02. LIE GROUP SIMILARITY TRANSFORMATIONS METHOD FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATION

Essential details of the Lie continuous point group similarity transformation method to reduce the number of independent variables of a nonlinear partial differential equation (NPDE) so as to obtain

respective ordinary differential equation (ODE) [05] is the following. Let the given NPDE in two independent variables *x* and *t* and one dependent variable *u* be

$$F(x,t,u,u_t,u_x,u_{tt},u_{xx},\dots)=0, \tag{2.01}$$

where $u_t, u_x \dots are$ all partial derivatives of dependent variables *u(x ,t)* with respect to the independent variable *t* and *x* respectively.

When we apply a family of one parameter infinitesimal continuous point Lie group transformations,

$$x = x + \epsilon X(x,t,u) + O(\epsilon^2), \tag{2.02}$$

$$t = t + \epsilon T(x,t,u) + O(\epsilon^2)\ , \tag{2.03}$$

$$u = u + \epsilon U(x,t,u) + O(\epsilon^2) \quad , \tag{2.04}$$

then we get the infinitesimals of the variables *u*, *t* and *x* as *U, T, X* respectively and ϵ is an infinitesimal parameter. The derivatives of *u* are also transformed as

$$u_x = u_x + \epsilon[U_x] + O(\epsilon^2)\ , \tag{2.05}$$

$$u_{xx} = u_{xx} + \epsilon[U_{xx}] + O(\epsilon^2), \tag{2.06}$$

$$u_{tt} = u_{tt} + \epsilon[U_{tt}] + O(\epsilon^2)\ , \tag{2.07}$$

where $[U_x], [U_{xx}], [U_{tt}]$ are the infinitesimals of the derivatives u_x, u_{xx}, u_{tt} respectively. These are called first and second extensions and that are given by [05],

$$[U_x] = U_x + (U|u - X_x)u_x - X_u u_x^2 - T_x u_t - T_x u_x u_t\ , \tag{2.08}$$

$$[U_{xx}] = U_{xx} + (2U_{xu} - X_{xx})u_x + (U|uu - 2X_{xu})u_x^2 - X_{uu}u_x^3,$$

$$+ (U_u - 2X_x u_{xx} - 3X_u u_x u_{xx} - T_{xx}u_t - 2T_{xu}u_x u_t - T_{uu}u_x^2 u_t$$

$$-2T_x u_{xt} - T_u u_{xx} u_t - 2T_u u_{xt} u_x, \tag{2.09}$$

and

$$[U_{tt}] = U_{tt} + [2U_{tu} - T_{tt}]u_t - X_{tt}u_x + [U_{uu} - 2T_{uu}]u_t^2$$

$$-2X_{tu}u_x u_t - T_{uu}u_t^3 - X_{uu}u_t^2 u_x + [U_u - 2T_t]u_{tt} - 2X_t u_{xt},$$

$$-3T_u u_{tt} u_t - X_u u_{tt} u_x - 3X_u u_{xt} u_t \ . \qquad (2.10)$$

The invariant transformation requirements of given NPDE (2.01) under the set of above transformations lead to the invariant surface conditions,

$$T\frac{\partial F}{\partial t} + X\frac{\partial F}{\partial x} + U\frac{\partial F}{\partial u} + [U_x]\frac{\partial F}{\partial u_x} + [U_{tt}]\frac{\partial F}{\partial u_{tt}} + [U_{xx}]\frac{\partial F}{\partial u_{xx}} = 0. \qquad (2.11)$$

On solving above invariant surface condition (2.11), the infinitesimals *X*, *T*, *U* can be uniquely obtained, that give the similarity groups under which the given NPDE (2.01) is invariant. This gives

$$T\frac{du}{dt} + X\frac{du}{dx} - \frac{du}{dU} = 0 \quad , \qquad (2.12)$$

The solutions of (2.12) are obtained by Lagrange's condition,

$$\frac{dt}{T} = \frac{dx}{X} = \frac{du}{U}. \qquad (2.13)$$

This yields,

$$x = x(t, C_1, C_2) \qquad (2,14)$$

and

$$u = u(t, C_1, C_2) \qquad (2.15)$$

where C_1 and C_2 are arbitrary integration constants and the constant C_1 plays the role of an independent variable called the similarity variable *s* and C_2 that of a dependent variable called the similarity solution $u(s)$ such that exact solution of given NPDE (2.01) becomes

$$u(x, t) \ = \ u(s) \ . \qquad (2.16)$$

On substituting (2.16) in given NPDE (2.01) that reduced to an ordinary differential equation (ODE) with *s* as independent variable and *u*(*s*) as dependent variable. Then all solutions u(s) of this reduced ODE also solutions of given PDE (2.01) through (2.16) .

03. LIE GROUP SIMILARITY TRANSFORMATION OF DShG EQUATION .

Then the invariant surface condition (2.11) of DShG equation (1.03) is

$$[U_{tt}] - [U_{xx}] - 8C\ U\cosh(u) + 16C\ U\cosh(u/2) = 0\ , \qquad (3.01)$$

where $[U_{tt}]$ and $[U_{xx}]$ are the second extension of u_{tt} and u_{xx} given by (2.09) and (2.10) Then equate various coefficients of partial derivatives of u as zero , we get the following constrained equations,

$$U_{tt} - U_{xx} - 8C\ U\cosh(u) + 16CU\cosh(u/2) = 0,$$

$$U_{xu} + U_{tt} = 0, \quad X_{xx} - 2X_{xu} - X_{tt} = 0, \quad T_x - X_t = 0, \quad 2U_{tu} - T_{tt} + T_{xt} = 0,$$

$$T_{xu} - X_{tu} = 0\ , \quad X_x = X_u = X_x = X_u = 0, \quad T_t = T_u = 0, \quad U = 0 \qquad (3.02)$$

On solving above constrained equations , we get

$$X = C\,t + v \quad , \tag{3.03}$$

$$T = C\,x + k, \tag{3.04}$$

$$U = 0. \tag{3.05}$$

,

The Lagrange's conditions (2.13) gives

$$(dx/X) = (dt/T) = (du/U) = 0, \tag{3.06}$$

leads to the following similarity variable s(x, t) , ..

$$s(x,t) = [\ \frac{C}{2}\ (x^2 - t^2)\ + (kx - vt) + (k^2 - v^2)/2C\ ] \tag{3.07}$$

Where C, k and v are arbitrary integration constants . Means ,we can transform the given second order nonlinear partial differential equation DShG (1.03) to a second order nonlinear ODE with the independant variable as the above similarity variable s , so that the exact solution of DShG equation (1.03) as

$$u(x,t) = u(s)\,. \quad . \tag{3.08}$$

Substitute (3.08) in the DShG equation (1.03), we get the similarity reduced ODE equation of (1.03) as

$$s\ \frac{d^2u}{ds^2} + \frac{du}{ds} = 8C\ \sinh(u) - 32C\ \sinh(u/2)\ ]\ . \tag{3.09}$$

The exact solution of above ODE (3.09) is

$$u(s) = -4\ \text{arc}\tanh(4\sqrt{-s})\ . \tag{3.10}$$

From (3.07), (3.08) and (3.10) we get the exact solution of DShG equation (1.03) as

$$u(x,t) = -\ 4\ \text{arc}\tanh(\ 4\sqrt{-s}\ )\ , \tag{3,11}$$

where s(x,t) is the (3.07) . Or

$$u(x,t) = -\ 4\ \text{arc}\tanh\{\ 4\ \sqrt{-[\tfrac{C}{2}(x^2 - t^2) + (kx - vt) + (k^2 - v^2)/2c]}\ \}\ , \tag{3.12}$$

is the exact solution of the DShG equation (1.03).

04. DISCUSSION

Above exact solution is complex valued and depends on three parameters C,k and v . In which $C \neq 0$. When k = v = 0 , we get exclusively hyperbolically rotational invariant new type of solution of DShG equation. Since $C \neq 0$, exclusively translational invariant and travelling wave type solution is not possible for DShG equation, but complete solution (3.12) is both rotationally invariant as well as translational invariant new type of exact solution of DShG equation. Both solutions coexist in the dynamics of DShG equation, hence one can study their interactions. Such type of pair of solutions ,this author reported for Liuoville equation, SU(2) Yang-Mills field equations and Double Sine Gordon equations . All known solutions of Klein-Gordon family of equations are of exclusively travelling wave types alone.

Corresponding to the three integration constants or parameters C, k and v of equations (3.05), (3.06) and (3.07) , we get three Lie Group generators [05] as follows,

$$G_1 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}\ , \qquad (4.01)$$

$$G_2 = \frac{\partial}{\partial x}, \qquad (4.02)$$

and

$$G_3 = \frac{\partial}{\partial t}\ . \qquad (4.03)$$

These generators form following Lie algebras [05] as

$$[\ G_1\ ,\ G_2\ ]\ =\ -\ G_3\ , \qquad (4.04)$$

$$[G_1\ ,\ G_3\ ]\ =\ G_2\ , \qquad (4.05)$$

and

$$[G_2\ ,\ G_3]\ =\ 0\ . \qquad (4.06)$$

Above (1 + 1) dimensional exact solution (3.12) of DShG equation (1.03) can easily be extended to

(3 + 1) dimensional ,then the similarity variable s(x,t) of (3.07) becomes,

$$s(x,t) = \frac{C}{2}[\ x^2+y^2+z^2-t^2) + [\ (k_1x+k_2y+k_3z) - (v_1t+v_2t+v_3t\ )\ ] + [\ (k_1^2 + k_2^2+k_3^2)-(v_1^2+v_2^2+v_3^2)]/2C \qquad (4.07)$$

where $k_{1,}(k_2,k_3\ ,\ v_{1,}\ v_2,v_3$ are integration constants..

This type of pair of solutions , this author reported for SU(2) Yang-Mills field equations[06],Liouville's equations[07] and Double sin-Gordon equations [08] , they form a new family of exact solutions

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