# Fair Division: Corrected Proportional Rule

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**DOI: 10.26821/IJSHRE.13.08.2025.130602**

## ABSTRACT

*In this paper, we intend to present and apply the Corrected Proportional rule (PCo) as a fair division rule in its own right, which constitutes with the methods PRRC (Resources Distribution Process based on the results of the Clustering) and PRRG (Resources Distribution Process at Group level) the three main methods of the Fair Division Approach based on Clustering and Reduction of inequalities (APCR).*

*The PCO rule reduces inequalities between individuals to a single level, that of the population considered as a single class. The originality of this paper is due to the fact that PCo rule is a fair division rule in its own right because of its results and is not used at all in the literature.*

**Keywords: Corrected proportional rule, Fair division, Divisible resources, Clustering, Réduction if inequalities, Process, Method.**

## 1. INTRODUCTION

Different methods of fair division of divisible resources exist. This is the case of the Proportional rule, the Adjusted Proportional rule. These methods are based on the demands or claims of individuals and the nature of the divisible resource.

There is reason to go beyond this approach by considering the case of several variables taking into account the type (homogeneous or heterogeneous) of the variables and the origin (contribution or not) of the resource [14]..

In the case where the resource is constituted from the contributions of individuals, the division rules above can apply directly but otherwise, it is fair to consider reducing inequalities between individuals because no one who contributed to create the resource. However, the reduction of inequalities cannot be done arbitrarily but between individuals who are similar or between those belonging to the same population [14], [15]. This uses the notion of Clustering to find individuals who are similar in order to reduce inequalities between them through solidarity. The same for those belonging to the same population. This is the Fair Division Approach based on Clustering and Reduction of Inequalities (APCR, from French "Approche de Partage equitable des ressources based sur la Classification et la Réduction des inégalités") [16].

The Corrected Proportional Rule (PCo, from French "règle Proportionnelle Corrigée") forms with the methods PRRC (from French, "Procédé de Repartition des resources à partir des Résultats de la Classification") and PRRG (from French, "Procédé de la Répartition des Resources au niveau des Groupe") the three main methods of the APCR. We will attempt to present it and apply it in the following.

We want to solve the problem of sharing a fixed homogenous divisible resource, such as money, in the case of problems taking into account the plurality and types of variables and those where beneficiary individuals have not contributed to creating the resource to be shared but who consider themselves to belong to the same population confused with the single class.

The plurality and types of variables problems were used in literature: Moreno-Ternero Juan D. (2006) [13] used several homogeneous variables that are claims of individuals to prove that among the existing bankruptcy rules, only the proportional rule produces the same result in the case where the claims of individuals related to different problems (salary, social benefits, equipment, travel, etc.) are aggregated (summed up) and in the case where the share of each individual is calculated from his claim related to each problem and then these shares related to these different problems are added up. For their part, Sheikhmohammady Majid and Madani Kaveh (2008) [20] in a problem of equitable sharing, more precisely, that of bankruptcy, used two heterogeneous variables "oil" (expressed in barrels) and "gas" (expressed in cubic feet, approximately 1/35 of a cubic meter) on the basis of which they answered the question on how to determine an equitable allocation which is associated with the legal status of the Carpal Ocean which is a multinational water resource. They converted the units of measurement of these two variables into a single one: dollar, because they knew the prices of a barrel of oil and a cubic foot of gas.

However, regarding clustering and reduction of inequalities in a fair division problem, they seem to us not to be used in literature. Most authors limit themselves to the application of the classification in different areas of life [6], to dividing individuals into different classes, others go further to the level of interpretation of the results by calculating the parameters (mean, standard deviation) of the classes resulting from the AHC (Ascending Hierarchical Clustering), to the determination of the most typical individuals (paragons and extremes) [8].

# 2. MATERIALS AND METHOD

## 2.1 Fair division

### *2.1.1 Fair division of a resource*

**Definition 1. 1 (Fair division problem).** Let be a vector w =($w_1$, . . . , $w_i$, . . . , $w_n$) ∈ $\mathbb{R}^n_+$ of the respective claims (values) of the n agents 1, 2, . . . , i, . . . , n belonging to the set I and a divisible resource C ∈ $\mathbb{R}^+$. We call the fair division problem the triplet (I, C, w) whose solution is a vector of individual shares c = ($c_1$, . . . , $c_i$, . . . , $c_n$) ∈ $\mathbb{R}^n_+$ with $\sum_{i=1}^{n} c_i = C$ [3], [9], [10]. In the case where C < $\sum_{i=1}^{n} w_i$, we speak of a deficit while when C ≥ $\sum_{i=1}^{n} w_i$, we are in the case of a surplus.

**Definition 1. 2 (Bankruptcy problem).** Let be a vector w = ($w_1$, . . . , $w_i$, . . . , $w_n$) ∈ $\mathbb{R}^n_+$ of the respective claims of the n agents 1, 2, . . . , i, . . . n belonging to the set I and a divisible resource C ∈ $\mathbb{R}^+$. A bankruptcy problem is a particular problem of sharing (I, C, w) satisfying the condition $0 \leq C \leq \sum_{i=1}^{n} w_i$ [20]. Any allocation in such problem is a n-tuple c = ($c_1$, . . . , $c_i$, . . . , $c_n$) ∈ $\mathbb{R}^n_+$ satisfying the following two properties :

1) $\sum_{i=1}^{n} c_i = C$ (Efficiency)

2) $0 \leq c_i \leq w_i$ (Reasonableness)

A classic bankruptcy situation consists of a certain sum of currency (resource) which must be shared between a few claimants (individuals) who have claims (values of variables) on the resource and the sum of claims is greater than the resource [17], [18].The division of resources, whether in the case of bankruptcy or any other case, must be done according to principles of justice.

### *2.1.2 Principles of justice*

As for the principles of justice, Forsé Michel and Parodi Maxime (2007) [7] allude to the three principles of justice : 1) absolute equality (or principle of equality) which assigns to the beneficiary individuals an equal share to each, 2) equity (or principle of merit) which shares the resource according to the merit of each individual proportionally to their merit, 3) the satisfaction of needs (at least the basic ones) (or principle of need) which allocates the resource according to the needs of each person. The PCo rule uses the principle of multidimensional reduced equity because it shares the resource from several values proportionally to the total value (sum of values) of each but after reduction of inequalities.

### *2.1.3 Fair division methods for divisible resources*

We consider the case of fair division methods concerning resources such as Currency (sum of money), electoral seats between candidates, common benefit, assets of a bankrupt company, overall cost of common equipment, etc. [5], [9], [23], [24]. These resources are variable (Case of overall cost of common equipment) or fixed (Case of a sum of money).

The existing fair division methods for fixed divisible resource include : 1) The proportional method/rule which distributes the resource proportionally to the demands of individuals [9]. 2) The equal surplus or rights-egalitarian method. 3) The uniform losses method or constrained equal loss method/rule. 4) The uniform gains or constrained equal award method. 5) The contested garment method [9] with its extensions which are : (1) the random-priority or random arrival method [1], [2]; (2) the Talmud method [2]; (3) the adjusted proportional rule [12], [20]. 6) The concede-and-divide rule [19], [23].

All these methods constitute the fair division approach, mainly taking into account the nature of the resource (divisible or indivisible) and the values (demands or claims) of individuals. The APCR approach from which the Corrected Proportional (PCo) rule is drawn takes into account not only the claims of individuals and the (divisible) nature of the resource but also its origin (non-contribution), the plurality (two or more) and the type (homogeneous or heterogeneous) of variables. It leads to the realization, particularly, of the clustering and the reduction of inequalities [14], [15].

## 2.2 Problem of transformation of the initial data

This section concerns the homogeneous and heterogeneous variables as used in Statics. Heterogeneous variables are those which are expressed on different units of measurement. Otherwise, they are said to be homogeneous.

Before carrying out the clustering or applying the sharing rule, it is necessary to check whether the variables are homogeneous or heterogeneous [14]. Clustering carried out without taking into account the problem of prior data transformation can lead to false results according to Mputu Denis-Robert (2022) [14].

In the case where they are homogeneous and for which the same unit of measurement is expressed in a unique way, we maintain the initial data and use them directly. If the variables are homogeneous and for which the same unit of measurement is expressed differently for each variable, we proceed by converting these different expressions into a single expression. And finally, if the variables are heterogeneous, we transform the initial data into centered data or reduced data or even centered-reduced data. More particularly, the transformation of the initial data into reduced data resolves, according to Mputu Denis-Robert (2022) [14], the problem of injustice in the distribution of resources due to the direct use of heterogeneous variables [14], [15].

The data are centered to bring the origin of the axes to the center of gravity (or barycenter) of the cloud of individuals (Case of homogeneous or heterogeneous variables) for better visibility of the graphic representation while they are reduced in order to annihilate the influence of units of measurement (Case of heterogeneous variables).

The transformation of data $X_j$ into reduced data $t_j$ is done using the following formulas [4]:

$$t_j = \frac{x_{ij}}{\sigma_j} \quad \text{(Reduced variable)}$$

(1.1)

where

$$\sigma_j = \sqrt{\frac{\sum_{i=1}^{n}(x_{ij}-\bar{X}_j)^2}{n}}$$

(1.2)

is the standard deviation of the variable $X_j$ and

$$\bar{X}_j = \frac{\sum_{i=1}^{n} x_{ij}}{n}$$

(1.3)

the arithmetic mean of the values checked by the variable $X_j$.

## 2.3 Clustering

Clustering is a method of Statistics which consists of finding classes (or clusters) which are such that the individuals of the same class are as similar as possible while those of different classes are the most dissimilar.

We distinguish between classic clustering and fuzzy clustering. Among the classic clustering methods, there are hierarchical methods and non-hierarchical methods. For the first, the classes are emerged from a hierarchy of partitions in an ascending manner (Ascending Hierarchical Clustering (AHC)) or in a descending manner (Descending Hierarchical Clustering (DHC)). While for the latter, the classes emerged from a hierarchy of partitions. This is the case of the k-means method.

### *2.3.1 Ascending Hierarchical Clustering (AHC)*

Carrying out the AHC involves the following steps [14]:

- Constitution of the data table,

- Calculation of the distances between individuals in pairs using a distance, for example the euclidean distance:

$$d(X_1, X_2) = \sqrt{\sum_{j=1}^{m}\left(X_{1j} - X_{2j}\right)^2}$$

(1.4)

- Calculation of the distances between groups of individuals (a group can be made up of a single individual) using an aggregation index, for example the centroid method (distance between the centers of gravity of the groups):

$$d(I_1, I_2) = d(G_1, G_2)$$

(1.5)

where $G_1$ and $G_2$ are the barycenters respectively of groups $I_1$ and $I_2$

- cutting of the dendrogram

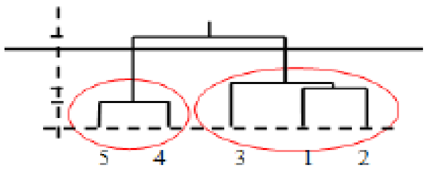


Fig 1 : Example of dendrogram cut into two classes

- interpretation of the AHC results.

### *2.3.2 k-means method*

The k-means method is a non-hierarchical clustering method which consists of starting from k initial centers of classes each made up of a single individual and resulting in k classes of individuals without establishing hierarchical links in the groupings of individuals. For more information see [22].

It is carried out by the following steps: 1) Choose a metric for calculating distances, define a number k of clusters (groups) and randomly choose k individuals as initial centers of the classes (groups). 2) Calculate the distances between each remaining individual and each initial center then assign each individual to the center that is closest to it. We find k clusters corresponding to the k chosen centers. 3) Determine the barycenters of these k clusters which constitute the k new centers and calculate the distances between these and all the individuals then assign each individual to the center (barycenter) which is closest to it and form k clusters comprising the individuals which are closest to them. 4) Stop the algorithm if the centers have remained sufficiently stable or if the intra-cluster variance stops decreasing or the inter-class variance stops increasing, otherwise continue by repeating the procedure from step 3).

### *2.3.3 Clustering and reduction of inequalities*

Speaking of clustering [21], considering the cut of the dendrogram, three possibilities present themselves: (1) the case where each individual belongs to its own class, (2) the case where the individuals are distributed in classes containing one or more individuals and (3) one where all individuals form a single class merged with the population.

Put in the context of the APCR which uses the notions of clustering and the reduction of inequalities [16], the PRRC and PRRG methods are suitable in the case where individuals are distributed in different classes containing one or more individuals while the PCo rule which concerns the case where all individuals form a single class confused with the population.

The PCo rule does not require the realization of the clustering due to the fact that individuals are assumed to all belong to a single class corresponding to the same population. Therefore, if the possible problem of transforming the initial data is resolved, we directly attack the problem of reducing inequalities at a single level, due to the fact that individuals belong to the same class combined with the population.

The reduction of inequalities is made using the following formula proposed by Mputu Denis-Robert (2022) [14], [15], [16] :

$$J_M = \frac{\sum_{i=1}^{n}(w_i - w_1)}{\sum_{i=1}^{n} w_i} \quad (1.6)$$

where n is the number of individuals in the population,

$$w_1 = min_{1 \le i \le n}\{w_i\} \quad (1.7)$$

the minimum of $(w_i), 1 \le i \le n$ (at the population level) with $w_1 \le \cdots \le w_i \le \cdots \le w_n$.

This reduction of inequalities index is important for measuring and reducing inequalities between those who are most similar or belong to the same population.

## 3. RESULTS

### 3.1 Corrected Proportional rule (PCo)

In this fair division rule, we admit that all the individuals belong to a single class comparable to the case of the unique partition found in the case where the cut of the dendrogram produces a single class containing all the individuals, a class which merges with the population. In this case, all individuals are dependent on each other and therefore become united in the population.

As for the reduction of inequalities, it is done at a single level : the single class (and therefore the population). The PCo rule allocates proportional shares after reducing inequalities to one level, due to the fact that all individuals belong to the single class combined with the population. In other words, the PCo rule produces the shares of individuals proportional to their corrected values $Z_i$ calculated using the inequality reduction index ($J_M$).

From the above, due to the fact that all individuals form a single class combined with the entire population, there is no question of still trying to determine the classes of individuals. From the point of view of clustering, the PCo rule in a classic environment and the PCo rule in a fuzzy environment (of clustering) merge. So classic PCo and fuzzy PCo are equal. The degree of belonging of each individual to this unique class is both in the classic clustering and in the fuzzy clustering. The application of the PCo rule therefore does not require the prior application of the clustering.

The steps for producing the PCo rule are as follows : 1) Presentation of the table of data maintained, converted or reduced, 2) calculation of the total values of individuals and the overall value, 3) calculation of the inequality reduction index at the level of the population (as a single class), 4) calculation of the corrected values of individuals in the population using the inequality index, 5) calculation of the shares of individuals proportional to their respective corrected value in the population.

### *3.1.1 Presentation of the table of maintained, converted or reduced data*

Let the resource C be shared between n individuals $x_1, \dots, x_i, \dots, x_n$ (or 1, . . . , i, . . . , n) of I from the values of m variables (with units of measurement maintained, converted or transformed (reduced)) $y_1, \dots, y_j, \dots, y_m$ and verified on the n individuals for which $y_{ij}$ is the value checked by individual $x_i$ for the variable $y_j$. Suppose they are presented in an array T(n,m ). We have :

$$T(n,m) = (y_{ij}) \text{ with } 1 \le i \le n, 1 \le j \le m \quad (2.1)$$

### *3.1.2 Calculation of individuals total values and the population global value*

We calculate for each individual i, its total (starting) value $w_i$ and the population global V :

$$w_i = \sum_{j=1}^{m} y_{ij}, \quad \forall\, i \in I \quad (2.2)$$

$$V = \sum_{i=1}^{n} w_i \quad (2.3)$$

### *3.1.3 Calculation of the inequality index at the population level*

The inequality index proposed by Mputu (2022, 2023, 2024) for the entire population is calculated from the individuals values.

$$J_M = \frac{\sum_{i=1}^{n}(w_i - w_1)}{\sum_{i=1}^{n} w_i} \quad (2.4)$$

or

$$J_M = \frac{\sum_{i=1}^{n}(w_i - w_1)}{V} \quad (2.5)$$

where n is the number of individuals in the population,

$$w_1 = min_{1 \le i \le n}\{w_i\} \quad (2.6)$$

the minimum of $(w_i), 1 \le i \le n$ (at the population level) and V the global value of the population.

### *3.1.4 Calculation of corrected values at the population level*

The corrected value of an individual i proposed by Mputu Denis-Robert [14], [15], [16] is calculated by the formula

$$Z_i = w_1 + w_i J_M, \forall i \in I \quad (2.7)$$

The corrected values are divided into two parts : the first is made up of the rich (transfer the values and see them reduced) and the second of the poor (who receive the transferred values and see them increased).

The expression for calculating the gaps allowing the determination of rich and poor is

$$E_i = Z_i - w_i \quad (2.8)$$

If $E_i \le 0$, we note $E_i(-)$ to mean the lost value by individual i. He is said rich. If on the other hand $E_i > 0$, we note $E_i(+)$ to mean gained value by individual i. He is said poor [14].

*Theorem 3.1 (Corrected value of an individual [15].* Let $Y = \{y_1, \dots, y_j, \dots, y_m\}$ be the set of m variables; $I = \{x_1, \dots, x_i, \dots, x_n\}$, the set of n individuals;

$x_i = (y_{i1}, \dots, y_{ij}, \dots, y_{im})$ the vector of m values checked by individual i, the possible transformation of the data starting point having been carried out, then the corrected value of an individual i ∈ I of total value $w_i$ is also written :

$$Z_i = \frac{w_1 V + w_i (V - n w_1)}{V}, \forall i \in I \quad (2.9)$$

*Proof.* Knowing that

$J_M = \frac{\sum_{i=1}^{n}(w_i - w_1)}{V} = \frac{\sum_{i=1}^{n} w_i - \sum_{i=1}^{n} w_1}{\sum_{i=1}^{n} w_i} = 1 - \frac{nw_1}{V}$, we have $:Z_i = w_1 + w_i J_M = w_1 + w_i\left(1 - \frac{nw_1}{V}\right) = \frac{w_1 V + w_i (V - nw_1)}{V}$ ■

*Corollaire 3. 1 (Total value of an individual based on its corrected value [15].* The total value of an individual i based on its corrected value is written as following :

$$w_i = \frac{V(Z_i - w_1)}{V - nw_1}, \forall i \in I \qquad (2.10)$$

*Proof.* Simply pull $w_i$ into the previous equation (2.10) to arrive at this formula

### *3.1.5 Calculation of individuals' shares*

The matter is to calculate individuals shares using their corrected values at the population level.

#### *3.1.5.1 Formula for calculating shares by the PCo rule using its corrected value*

The share of an individual i proportional to his corrected value $Z_i$ is calculated by the formula :

$$c_i^{PCo} = \frac{Z_i}{V} C, \forall i \in I \qquad (2.11)$$

#### *3.1.5.2 Simplified formulas for calculating shares using the PCo rule*

*Proposition 3.1 (Corrected proportional rule)[15].* Let $W = (w_1, \ldots, w_i, \ldots, w_n)$ be a vector of respective total values (claims) of individuals 1, 2, . . . , i, . . . , n belonging to the set I; consider C, the resource to be shared between the n individuals of I and (I, C, W) a fair sharing problem, suppose that all the individuals form the same class. By the corrected proportional rule (PCo), we mean a sharing rule denoted $c^{PCo}$ which to any problem (I, C, W) and to any individual i ∊ I, we associate the part of individual i :

$$c_i^{PCo}(I, C, w) = c_i^{PCo} = \left(\frac{w_1}{V} + \frac{J_M}{V} w_i\right) C, \forall i \in I \qquad (2.12)$$

with $w_1$ the smallest total value, V the overall value (claim) and $J_M$ the inequality index proposed by Mputu Denis-Robert [14], [15].

*Proof.* We know that the corrected value of an individual i with total value $w_i$ is $Z_i = w_1 + w_i J_M$ and let $V = \sum_{i=1}^{n} w_i = \sum_{i=1}^{n} Z_i$. Considering an individual i, the share of a resource C of individual i proportional to its corrected value $Z_i$ is $\frac{Z_i}{\sum_{i=1}^{n} Z_i} C \Leftrightarrow \left(\frac{w_1 + w_i J_M}{V}\right) C \Leftrightarrow \left(\frac{w_1}{V} + \frac{J_M}{V} w_i\right) C$ ■

*Corollary 2.2 (Shares of individuals following the PCo rule).* Consider individuals 1, 2, . . . , i, . . . , n of respective claim $w_1, w_2, \ldots, w_i, \ldots, w_n$ who must share the resource C. Suppose that the individuals consider themselves each belonging to a class considered as a population in its own right. Then the shares of individuals are calculated according to the PCo rule using the formula (2.12)

*Theorem 3.2* (Corrected proportional share of an individual [14], [15]). The corrected proportional share of an individual i belonging to the set I = {1, . . . , i, . . . , n} is also determined using the following formula :

$$c_i^{PCo} = \frac{w_1 V + w_i (V + nw_1)}{V^2} C, \forall i \in I \qquad (2.13)$$

where, $w_i$ is the (notably corrected) value of an individual i at the population (or single class) level, $w_1 < w_i$, ∀i ≠ 1.

*Proof.* Just replace $J_M = \frac{\sum_{i=1}^{n}(w_i - w_1)}{\sum_{i=1}^{n} w_i}$ in the formula of the previous theorem ■

### *3.1.6 Function of corrected proportional shares*

*Proposition 3.2 (Function of corrected proportional shares).* The function of the corrected proportional shares for the PCo rule according to the reduction index $J_M$ is an affine function such that given the sets A' of the corrected total values (at the population level) of the individuals and B' corrected proportional shares, C being the resource to share, we have :

$$c^{PCo} : A' \subset \mathbb{R} \longrightarrow B' \subset \mathbb{R}$$

$$w_i \longmapsto c^{PCo}(w_i) = \frac{w_1}{V} C + \frac{J_M}{V} w_i C$$

where $w_1$ is the minimum of the distribution of total values at the level of the population considered as a single class, $J_M$ the index of reduction of inequalities.

*Remark 3.1* The function $c^{PCo}$ of the corrected proportional shares of individuals constitutes an affine function of angular coefficient $\frac{J_M}{V} C$. We call the coefficient $\frac{w_1}{V}C$, the proportional share of the poorest or least well-off individual.

In this approach, individuals grant to each the proportional share of the least well-off then they share the rest of the resource proportionally to the product $J_M w_i$.

### 3.1.7 PCo rule properties

*Proposition 2. 1 (PCo rule properties).* Let $W = (w_1, \dots, w_i, \dots, w_n)$ be the vector of the revendications (total values) of individuals, $w_i$ total value of i on the resource C, $Z_i$ its corrected value (" corrected revendication"), $c_i(\mathrm{I, C, W})$ (or $c_i(\mathrm{I, C, Z})$) the division results from the resource C between I individuals from their total values w (or from their corrected values Z), $c_i^{PCo}$=$c_i$ the share of i from the resource C from the division through the PCo rule. So, the latter checks the following properties :

1) *Equal treatment of equals*

$Z_i = Z_j \Longrightarrow c_i = c_j$, ∀i,j ∈ I.

The individuals with equal corrected values got equal shares.

2) *Efficiency*

$\sum_{i\in I} c_i = \mathrm{C}$

The resource must be shared in its entirety.

3) *Monotony*

*(1) Resource monotony*

∀I,C, C' et Z : si C ≤ C' ⟹ $c_i(\mathrm{I, C, Z}) \le c_i(\mathrm{I, C', Z})$, ∀i ∈ I.

If for two given resources one of them is bigger than the other, so the correspondant shares from the biggest are also bigger than the one of the smallest.

*(2) Monotony of claim*

∀ I, C et Z : si $Z_i \le Z_j \Longrightarrow c_i(\mathrm{I, C, Z}) \le c_j(\mathrm{I, C, Z})$,

∀i,j ∈ I.

If the corrected value of one individual is bigger than another, so his share will be also bigger.

*4) Dependence of individuals*

If I = {1,..., i,..., n} ⟹ c(I,C, Z) and I' = I ∪{i'} ⟹ c(I',C, Z).

Then, c(I,C,Z) ≠ c(I',C,Z)

Now I = {1,..., i,..., n} ⟹ c(I,C, Z) and I' = I\{i} ⟹ c(I',C, Z).

So, c(I,C,Z) ≠ c(I',C,Z).

Individuals are sensitives to the arrival or departure of an individual. So an individual arrival or departure modify the results of division.

*5) Ingratitude towards individuals with zero demands*

Si $Z_i = 0 \Longrightarrow c_i^{PCo} = 0$.

The individual who brings nothing to the population will have nothing. Each individual must contribute something to benefit also something. However, its presence is in favor of the rich and its absence in favor of the poor.

### 3.1.8 PCo rule algorithm

This algorithm which the complexity is quadratic will allow to calculate individuals shares using PCo rule for several variables in the case of a matrix with the software R [11].

***Enter*** *: Matrix X of several variables $X_1, \dots, X_j, \dots, X_m$ which the vectors of values respectively of individuals 1, . . . , i, . . . , n are $(x_{i1}, \dots, x_{ij}, \dots, x_{im})_{1\le i\le n}$.*

*Resource to share is C.*

***Output :*** *The shares $c_1^{PCo}$ , . . . , $c_i^{PCo}$ , . . . , $c_n^{PCo}$ of n individuals.*

***START***

```
ProCoHoM=function(X,C)
{
if (is.matrix(X))              # if X is a matrix
{
 if(C>=0)              # if C is a positive number ou null
{
if
 (length(as.vector(X[X>=0]))==length(as.vector(X)))        # if the length of the vecteur formed by termes of X which are positive or null is equal to the vecteur of all of the termes of X. If it'is so that means X all of the termes of X are positive or null.
 {
Wi=as.matrix(rowSums(X))       #The matrix of total values (claims) of individuals (claimers) or the sum of values of each of them
W1=min(Wi)          #Minimum of all of the total values
V=sum(Wi)          #Global value or sum of all of the values of individuals
```

```
D= sum(Wi-W1)        #Matrix sum of the differences between each total value and the smallest value
JM=D/V               #Indice of inequality proposed by Denis-Robert Mputu
Zi=(W1)+((Wi)*(JM))  #Matrix of corrected values of individuals
Ei=(Zi)-(Wi)         #Matrix of the difference between the correted values and the stanting values
partHoM=((Zi)*C)/sum(Zi)  #Matrix of shares with reduction of inequalities from C resource
Tot=sum(partHoM)
print("Results for the share using Corrected Proportional rule are:")
list("X matrix"= X, "Total values (claims) of individuals (Claimers)"=Wi, "The minimum of distribution is :"=W1, "Global value (sum of claims) of the population"=V, "Resource to be shared is"=C, "Mputu Inequality index"=JM, "Corrected values of individuals"=Zi, "deviation between corrected values and the starting totales values"=Ei , "Nota"= "In above matrix, individual with negative value is rich and this with the positive one is poor", "Individuals shares from the Corrected Proportional rule for a Matrix and homogeneous variables"=partHoM, "Total"=Tot)

}
else print("Sorry. Impossible to compute because one of the matrix termes at least is negative. Please write positive termes")
}
else print("Sorry. Impossible to compute because the right number is negative or the left object has a negative term. Please write positive numbers")
}
else print("Sorry. Impossible to compute because the left object is not a matrix or the right one is not a number. Please write the matrix at the left side and the number at the right one")
}
```

***END***

To apply this algorithm, it is enough to copy and paste it in R console and then: to consider a matrix X of individuals values and a number C as a resource, to write ProCoHoM=function(X,C) and to validate.

As for the complexity of this algorithm, it is polynomial (quadratic).

Indeed, it includes simple instructions such as addition, subtraction, multiplication, division of real numbers, and assignments with a constant complexity: O(a); complex instructions for the sum, product, and quotient of a single-row or single-column matrix and a real number containing a loop with a linear complexity: O(n); and a complex instruction for determining the minimum using sorting containing two loops with a quadratic (and therefore polynomial) complexity: $O(n^2)$. Summing all these complexities, we find $O(n^2)$. Therefore, the computational complexity of the Corrected proportional rule (PCo) function is quadratic. The computational time is therefore polynomial (quadratic).

## 3.2 Application

### *3.2.1 Problem Statement*

We give a scholarship of C=50MFC (Fifty million Congolese Francs) to five students a, b, d, e and f who obtained, out of 10 points in Maths and French, the following respective grades : (6, 4) , (7, 8), (2, 3), (8, 9), (2, 6). Knowing that 1 point obtained corresponds to a bonus of 1 MFC (One million Congolese Francs), we ask to calculate the shares of individuals using the Corrected Proportional rule (PCo) and determinate rich and poor.

### *3.2.2 Problem Focus*

Before calculating the shares, let us emphasize that the values verified by these five individuals come from two homogeneous variables which are “Points obtained in Maths” and “Points obtained in French”. There is therefore no need, for this case, to convert these variables to the same expression of the unit of measurement or to transform them into reduced variables. In addition, these students did not contribute to create the 50MFc resource that they must share. Which gives the idea of solidarity between the closest beneficiaries. Hence the importance of clustering and therefore of reducing inequalities between those belonging to the same class and/or the same population. As we said above, the PCo rule is not affected by the clustering (It is not so for PRRC and PRRG).

### 3.3 Solution by the Corrected Proportional (PCo) rule

- The resource to share is C=50MFC. The variables are homogeneous (Grades obtained in Math and in French) which do not require prior conversion or transformation. The vectors of the values of individuals a, b, d, e, f are respectively (6, 4), (7, 8), (2, 3), (8, 9), (2, 6).

- The respective total values of the individuals are :

$w_i = (a, b, d, e, f) = (10, 15, 5, 17, 8)$

with $w_1 = d = 5$

- The overall value is

$V = \sum_{i=1}^{5} w_i = 55$

- The inequality index is

$J_M = \frac{\sum_{i=1}^{5}(w_i - w_1)}{V} = \frac{5+10+0+12+3}{55} = \frac{30}{55} = 0.54545$

- The corrected values are:

$Z_i = w_1 + w_i J_M$=(a, b, d, e, f) =(10.46; 13.18; 7.73; 14.27; 9.36)

- Their corrected proportional shares are calculated by the formula

$$c_i^{PCo}(C) = \left(\frac{w_1}{V} + \frac{J_M}{V} * w_i\right) C = \frac{Z_i}{V} * C$$

For individual a, we have :

$c_a^{PCo}(50) = \left(\frac{5}{55} + \frac{0.54545}{55} * 10\right) * 50$

=0.19 ∗ 50 = 9.5 MFC

Thus, the shares of individuals by the PCo rule are :

$c_i^{PCo}(50) = (c_a^{PCo}, c_b^{PCo}, c_d^{PCo}, c_e^{PCo}, c_f^{PCo})$

= (9.5; 11.98; 7.03; 12.98; 8.51)

The rich and the poor are determined as follows:

$E_i = Z_i - w_i = (E_a, E_b, E_d, E_e, E_f)$

$= (10.46;\ 13.18;\ 7.73;\ 14.27;\ 9.36) - (10; 15;\ 5; 17; 8)$

=(0.46 ; -1.82 ; 2.73 ; -2.73 ; 1.36)

Individuals which have $E_i < 0$ are rich: b and e. Those with $E_i \geq 0$ are poor : a, d and f.

### 3.4. Solution by the Corrected Proportional (PCo) rule using R software

Using R software, it produces the same results. Simply proceed as follows: Copy and paste the PCo algorithm from above into the R console, then enter the matrix of values for individuals a, b, d, e, f, which are respectively (6, 4), (7, 8), (2, 3), (8, 9), (2, 6) and the resource to be shared C=50, then use the ProCoHoM function:

```
X=matrix(c(6, 4, 7, 8, 2, 3, 8, 9, 2, 6), nrow=5, byrow=TRUE)

C=50

ProCoHoM(X, C)
```

## 4. DISCUSSION

Consider the Proportional rule (P) with the same data:

$w_i = (10, 15, 5, 17, 8)$, V=55 and C=50. The shares of individuals are:

$c_i^P(50) = \frac{w_i}{V} * C = (9.09; 13.64;\ 4.55; 15.45; 7.27)$.

If we compare these shares with those found with PCo, we can remark that the individual with the smallest total value has 4.55 MFC in Proportional rule but 7.03 MFC in PCo rule and the one with the bigest total value has 15.45 MFC in Proportional rule but 12.98 MFC in PCo rule. To means PCo reduced inequality between individuals and is a fair division in its own right belongs to the family of APCR methods.

## 5. CONCLUSION

As conclusion, we have presented the PCo rule as a fair division method in its own right in the same way as the Proportional rule and Adjusted Proportional rule. It is part of the Fair Division Approach based on Clustering and Reduction of Inequalities (APCR) which methods are: PRRC which reduces the inequalities in two levels (in the classes or clusters and the population), PRRG which reduces the inequalities in one level (in the classes) and PCo which reduces the inequalities in one level also (in the population). These methods are preferred by the individuals with the smallest values. We used PCo with data of homogeneous variables and show how to determinate rich and poor individuals. It is also applicable in the case of heterogeneous variables.

About clustering, there is classical clustering and fuzzy clustering [25], [26]. The fuzzy clustering and reduction of inequalities will be used later. So we will write PRRC (or PRRC/D) and PRRG (or PRRG/D) for the case of the classical clustering and PRRC/F and PRRG/F for the fuzzy one.

## 6. ACKNOWLEDGMENTS

We thank professor SUMATA Claude of Université Pédagogique Nationale (UPN) and professor NGOIE Ruffin-Bénoît of Institut Supérieur Pédagogique de Mbanzangungu (ISP/Mbanzangungu) for their wise observations.

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