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Results on Harmonious Graphs

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ABSTRACT

This paper aims to focus on some of the results already in existence for the Harmonious Labeling of Graphs. The approach will be to summarize them in different flavor and possible different labeling. Several of these families of graphs were suggested by the application like consider a network of transmitting stations, each of which must be able to communicate with certain others-those to which it is linked in the network. The total bandwidth available is divided into e channels, where eis the number of links in the network, and each station x is assigned a number λ(x) from $\mathbb{Z}e$. When x and y communicate they use channel number λ(x) +λ (y). If the numbering is harmonious, each channel is assigned to exactly one link.Harmonious graphs are also interesting because they lead to modular versions of various combinatorial problems. For example, a harmonious labeling of the friendship graph Fnmay be regarded as a modular generalization of the Langford-Skolem Problem.

General Terms

Graph, Complete Graph, Bipartite Graph, Complete Bipartite Graph, Labeling of Graph, Join of Graphs.

Keywords

Caterpillar, Fan, Friendship Graph, Web, Harmonious Labeling of Graph, Harmonious Graphs.

1. INTRODUCTION

Discrete mathematics is a dynamic field in theory and applications. It concerns with structures defined on discrete sets. Researchers in discrete mathematics have established important connections with main stream areas of pure and applied mathematics and as a consequence, research techniques and problems are drawn from a wide range of different fields. Discrete mathematics has now burgeoned into one of the fastest growing branches of mathematics. This mathematics has various branches like mathematical logic, Boolean algebra, combinatory, finite state machine and graph theory.The present work deals with some interested topics in graph theory which has become more interesting due to its diversified applications. Interest in graph theory and its applications have grown exponentially in last century due to rise of computer age. Graphs play vital role in the establishment of physical network like electrical circuit, transportation network, telephone network etc. Graphs serve a better tool to explain eco system, sociological relationships and flow of control in a computer system.There are varieties of interesting fields for research in graph theory. Some of them are domination in graph, decomposition of graph, graph coloring, graph labeling. The present work is aimed to discuss Harmonious labeling of graphs. Investigations in the field of graph labeling is interesting because the field itself is rapidly emerging and a whole galore of seemingly related or even unrelated open problems provide motivation for advance research.

2. DEFINITIONS

2.1 Graph

A*graph* *G= (V,E)*consists oftwo finitesets*V*,thevertexsetofthegraphwhichisanon-emptysetofelementscalled*vertices*and*E*,theedgesetofthegraphwhich ispossiblyemptysetofelementscalled*edges*,suchthateachedgeein*E*isassigned anunorderedpairofvertices*(u,v)*calledthe*endvertices*of *e*

2.2 Isolated Vertex

A vertexofgraph*G*whichisnottheendofanyedgeiscalledan *isolatedvertex*.

2.3 Adjacent Vertex

Two vertices which arejoinedbyanedge aresaidto be*adjacent vertices*.

2.4 Complete Graph

A*completegraph*isasimplegraph(without loop or parallel edges)inwhicheachpairofdistinctverticesisjoinedbyan edge.ItisdenotedbyK_n

2.5 Bipartite Graph

Let*G*beagraph.Ifthevertexset*V*of*G*canbepartitionedint otwonon-emptysubsets*X*and *Y*insuchawaythateachedgeof*G*has oneendvertex in*X*andoneendvertexin*Y*than*G*iscalleda*bipartitegraph*. Thepartition*V=X∪Y*iscalleda*bipartition*of*G.*

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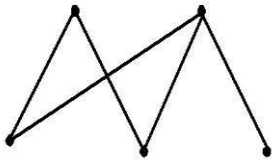

Fig 1: Bipartite Graph

2.6 Complete Bipartite Graph

A *complete bipartite graph* is a simple bipartite graph G, with thebipartition $V = X \cup Y$inwhich every vertexin$X$isjoinedto every vertexof$Y$.IfXhas$m$verticesand$Y$has$n$verticesthan suchagraphisdenotedby$K_{m,n}$.*Fig2*showsacompletebip artitegraph$K_{3,3}$

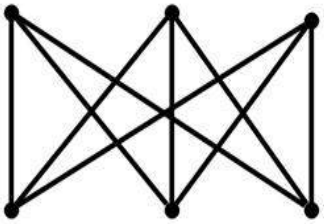

Fig 2: Complete Bipartite Graph

2.7 Walk

A*walk*inagraph$G$isafinitesequence$W=v_0e_1v_1e_2v_2...v_{k-1}e_kv_k$whosetermsarealternately vertices andedgessuchthateach edge$e_i$hasend vertices$v_{i-1}$and$v_i$for$1 \le i \le k$.

Thenumberofedgesina walkiscalledthe*lengthofthewalk*.

2.8 Path

Intheverticesv$_0$,v$_1$,v$_2$,,,,v$_k$ofthewalk$W=v_0e_1v_1e_2v_2...v_{k-1}e_kv_k$aredistinctthan$W$iscalleda*path*.Apath with$n$verticesisdenotedbyP_n.$W_4=v_1e_1v_2e_2v_5$isapathbut $W_5=v_1e_2v_2e_3v_5e_6v_1e_9v_3$isnotapath since$v_1$isrepeated. A path is called *closed path* if its end vertices are the same.

2.9 Cycle

Aclosedpathinagraphin whichallitsedgesaredistinctiscalleda*cycle*.$C_1=v_1e_2v_2e_5v_3e_9v_1$isacycleoflength3, but$C_2=v_1e_2v_2e_5v_3e_7v_4e_4v_2e_2v_1$isnotacycleasvertex$v_2$isre peated.

2.10 Tree

Agraph$T$iscalleda*tree*if itisaconnectedacyclicgraph (without any cycle).

2.11 Rooted Tree

A*rootedtree*isatreehavingadistinguishedvertex$r$calledt he*root*suchthatforeveryothervertex$v$,thereisan$r$–$v$path.

2.12 Caterpillar

A*Caterpillar*isatreewiththepropertythattheremovalof itspendantverticesleavesapath. In other words it is a tree that all pendent vertices are at distance 1 from the central path.

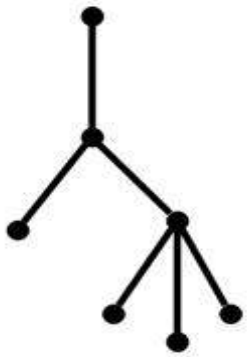

Fig 3: Caterpillar

2.13 Symmetric Tree

*Symmetricaltree*isarootedtreeinwhicheverylevelconta insverticesofthesamedegree.

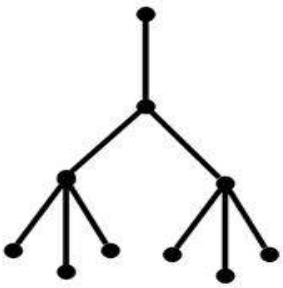

Fig 4: Symmetric Tree

2.14 Olive Tree

An*olivetree*isarootedtreeconsistingof$k$bracheswheret he ithbranchisapathof length$i$.

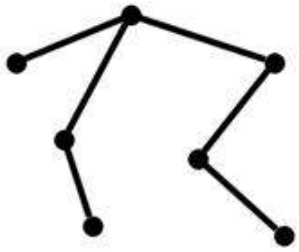

Fig 5: Olive Tree

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2.15 Star Graph

A *star graph* with *n* vertices is a tree with one vertex having degree *n –1* and other *n –1* vertices with degree one. A star graph with *n+1* vertices is denoted by $K_{1,n}$.

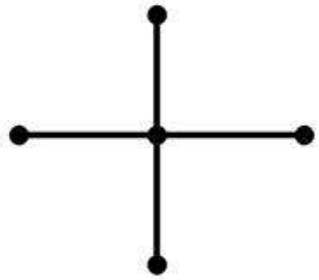

Fig 6: Star Graph

2.16 Banana Tree

A *banana tree* is a tree obtained from a family of stars by joining one end vertex of each star to a new vertex not include.

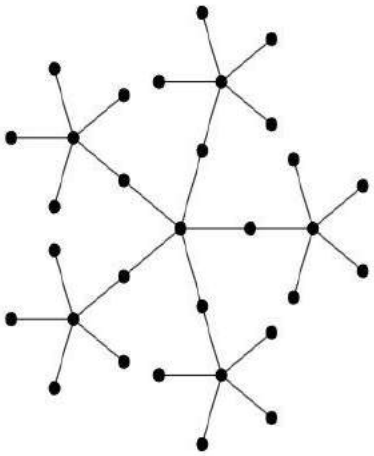

Fig 7: Banana Tree

2.17 Join of Graphs

Let G_1 and G_2 be two graphs with no vertex in common. We define the *join* of G_1 and G_2 denoted by G_1+G_2 to be the graph with vertex set and edge set given as follows:

$V(G_1+G_2)=V(G_1)\cup V(G_2), E(G_1+G_2)=E(G_1)\cup E(G_2)\cup J$

Where $J=\{x_1x_2 : x_1\in V(G_1), x_2\in V(G_2)\}$, i.e. edge set is consisting of those edges which are in G_1 and G_2 and the edges obtained by joining each vertex of G_1 to each vertex of G_2.

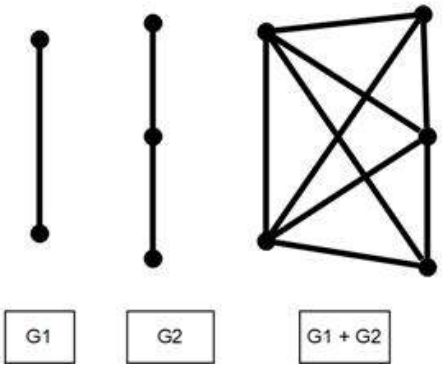

Fig 8: Join of Graphs

2.18 Wheel

The *wheel graph* W_n is defined to be the join of K_1 and C_n i.e. the *wheel graph* consists of edges which join a vertex of K_1 to every vertex of C_n.

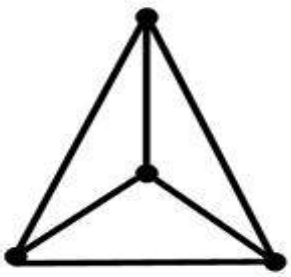

Fig 9: Wheel W_3 Graph

2.19 Helm

The *helm* H_n is a graph obtained from a wheel by attaching a pendant vertex at each vertex of the *n–* cycle as shown in the *Figure 10*.

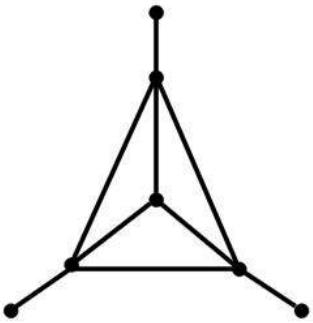

Fig 10: Helm

2.20 Web

A *web* graph is the graph obtained by joining the pendant vertices of a helm to form a cycle thanadding a single pendant edge to each vertex of this outer cycle.

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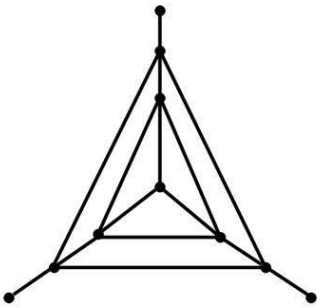

Fig 11: Web

2.21 Flower

A*flower*isagraphobtainedfromahelmbyjoiningeachpendantvertextothecentralvertexofthehelm.

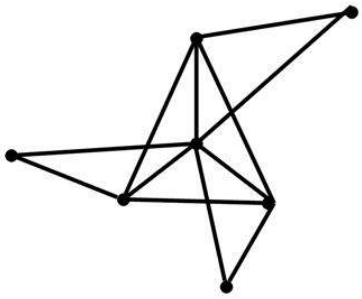

Fig 12: Flower

2.22 Product Graph

The*productofgraphsG1*and*G2*denotedby*G1xG2*isthegraphwithvertex set*V(G1)xV(G2)={(u,v)/u∈V(G1)andv∈V(G2)}*and*(u,v)*adjacentto*(u',v')*ifandonlyifeither*u=u'*and*vv'∈E(G2)*or*v=v'*and*uu'∈E(G1)*.

2.23 Ladder

The*ladderLn* (n≥ 2)istheproductgraph*P2 xPn* whichcontains*2n*verticesand*3n–2*edges.

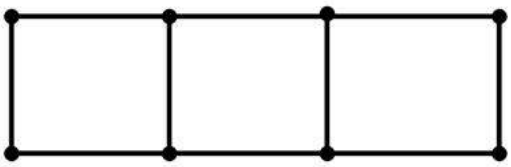

Fig 13: Ladder

2.24 Friendship Graph

A*friendshipgraphFn*isagraphwhichconsistsof*n*triangleswithacommonvertex.Friendshipgraph*F4*isshowninthefollowing*Fig 14*

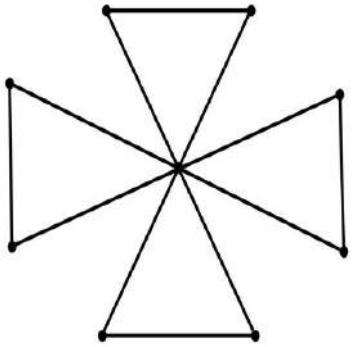

Fig 14: Friendship Graph

2.25 Fan

The*fanfn(n≥ 2)* is obtainedby joiningall verticesof*Pn*toa further vertexcalledthe*center*and contains*n+1vertex*and*2n-1*edges.i.e.*fn=Pn+K1*.Fan*f4*isshowninFig 15.

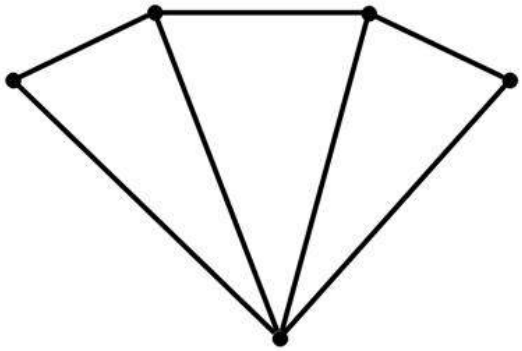

Fig 14: Fan

2.26 One Point Union of Two Graph

This is denoted by $K_n^{(2)}$ (n ≥ 3) and it consists of two copies of K_n sharing a common vertex and contains 2n – 1 vertices and n(n – 1) edges. Fig 15 shows $K_4^{(2)}$

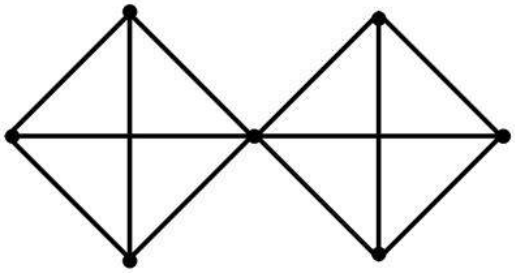

Fig 15: $K_4^{(2)}$

2.27 The Graph g_n

Thegraph*gn*isagraphwith*n+2*verticesand*3n–1*edgesobtainedbyjoiningallverticesof*Pn*totwoadditionalvertices.*Fig 16*showsagraphg_6.

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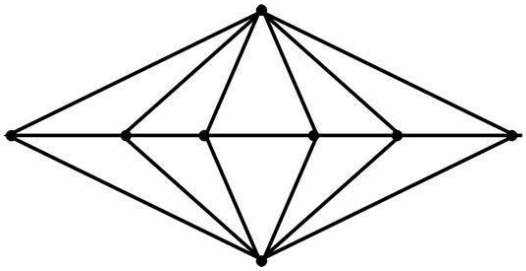

Fig 16: g_6

2.28 Harmonious Labeling of Graph

Harmonious labeling for a graph G with v nodes and e edges (e ≥ v) is a labeling technique in which we label x vertices with label λ(x) of $\mathbb{Z}e$ in such a way that resulting edge labeling λ(x) + λ(y) for edge xy is distinct.

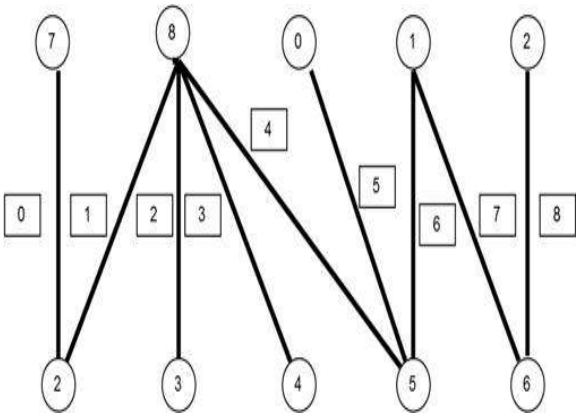

Fig 18: Harmonious Labeling for Caterpillar

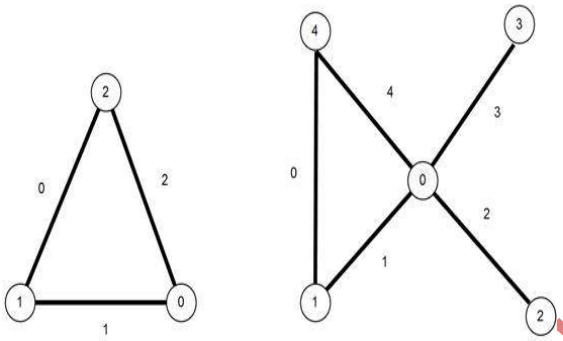

Fig 17: Harmonious Labeling of Graphs

2.29 Harmonious Graph

A connected graph which admits harmonious labeling is called *harmonious graph*.

3. Results on HARMONIOUS LABELING

3.1 The vertices labels of Harmonious Graph are distinct.

3.2 Edges labels have to be from 0, 1, 2, 3,,,, , e-1for $\mathbb{Z}e$

3.3 Caterpillars are Harmonious.

After drawing caterpillar as a bipartite graph, having l nodes on left and r nodes on the right.So there are e=l+r-1 edges. If e is odd or if e is even and r is odd, choose a ∈ $\mathbb{Z}e$so that 2a = r – 1 in $\mathbb{Z}e$. If e and r both are even, then l is odd and we choose a so that 2a = 1 – l. We then label the left side vertices as a, a+1, ,,, a + l -1 and right side vertices as –a, 1-a,,,, r -1 – a. By doing so we can achieve the full set of labels from {0, 1, 2, ,,,, e-1} with a repeated or – a repeated.

3.4 The complete graph with v nodes is harmonious if and only if v ≤ 4.

The below figure represent the complete graphs K_2, K_3 and K_4.

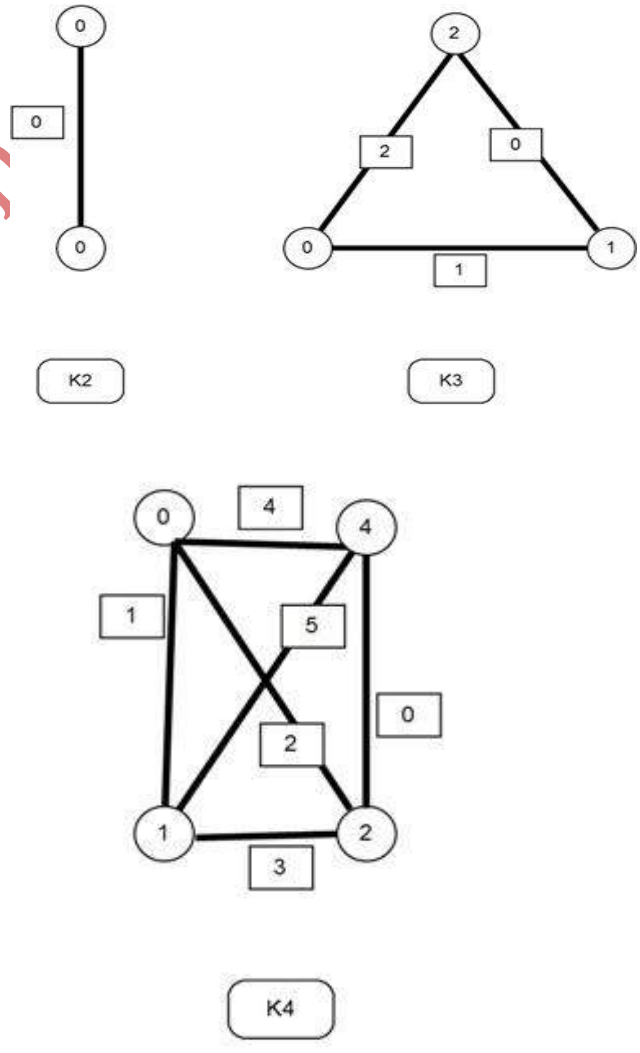

Fig 19: Harmonious Labeling for K_2, K_3 and K_4

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3.5 All Trees nodes n ≤ 9 vertices are Harmonious.

3.6 Cycle Cn (n ≥ 3) is Harmonious if and only if n is odd.

We label the vertices with 0, 1, 2, ,,, n-1 and then obtain the corresponding edge labeling. The figure below gives harmonious labeling of C_3, C_5 and C_7.

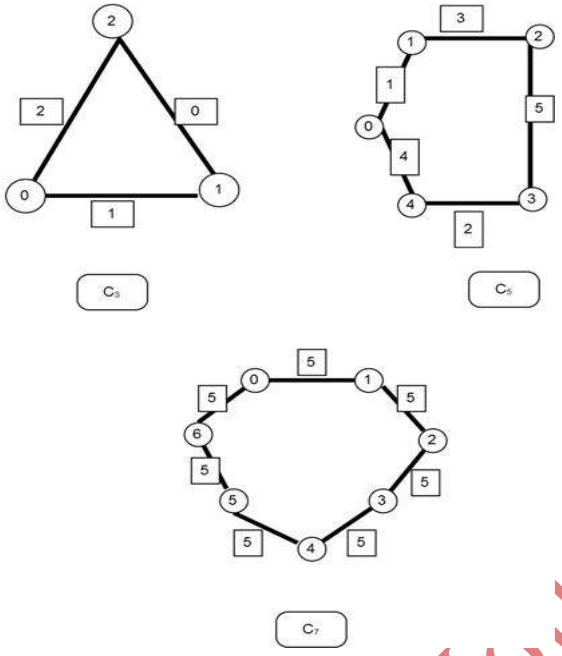
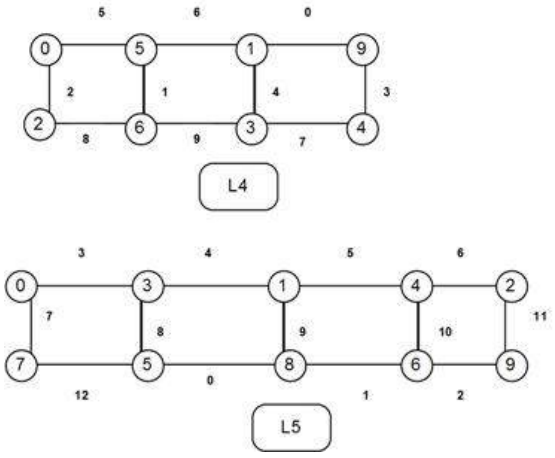


Fig 19: Harmonious Labeling for C_3, C_5 and C_7

3.7 All Ladders are Harmonious except L2.

L2 is not harmonious since L2 is a cycle C4 and which is not harmonious (3.6 above). Some Harmonious labeling of ladders L_4, L_5, L_6, L_7 and $L_8$are as shown in the figure below.



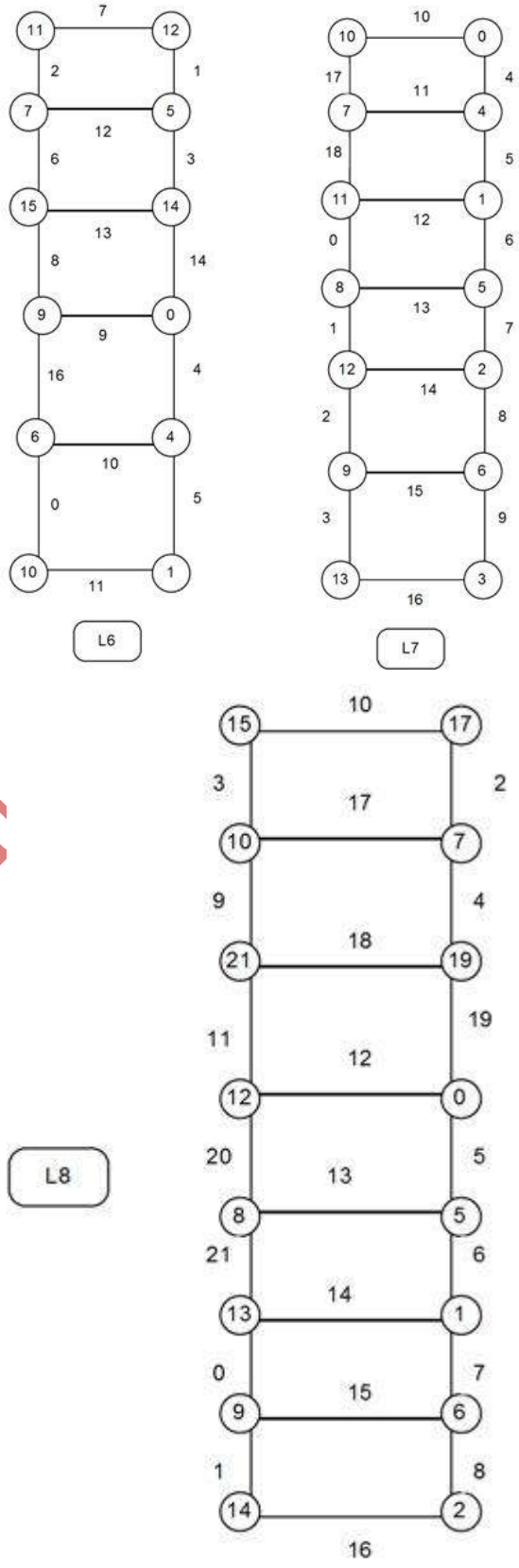


Fig 20: Harmonious Labeling ofL_4, L_5, L_6, L_7 and L_8

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3.8 The repeated vertex in the case of trees can be any vertex of $\mathbb{Z}_e$

This fact can be verified by the following two figures.

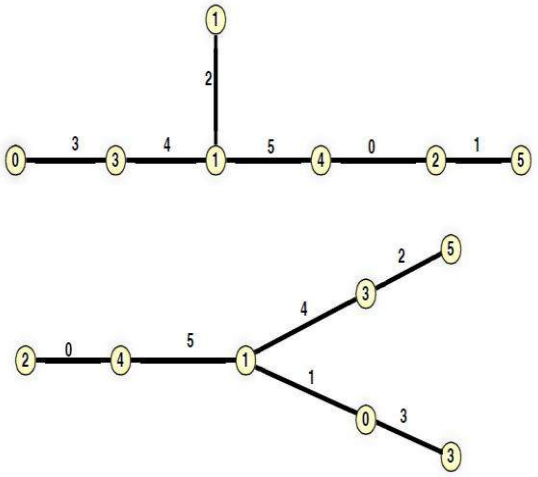


Fig 21: Harmonious Labeling of trees with different repeated vertex

3.9 If f is a harmonious labeling of G with e edges than af(x) + b is generates another harmonious labeling for G, where a is invertible element of $\mathbb{Z}_e$ and b is any element of $\mathbb{Z}_e$.

This result again can be verified from below two figures.Here we are choosing a = 5 and b = 4

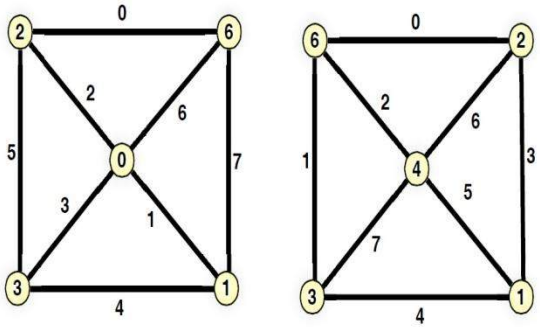


Fig 22: Harmonious Labeling of W_4 and applying 5f(x) + 4

3.10 Friendship graph is Harmonious except n ≡ 2 (mod 4)

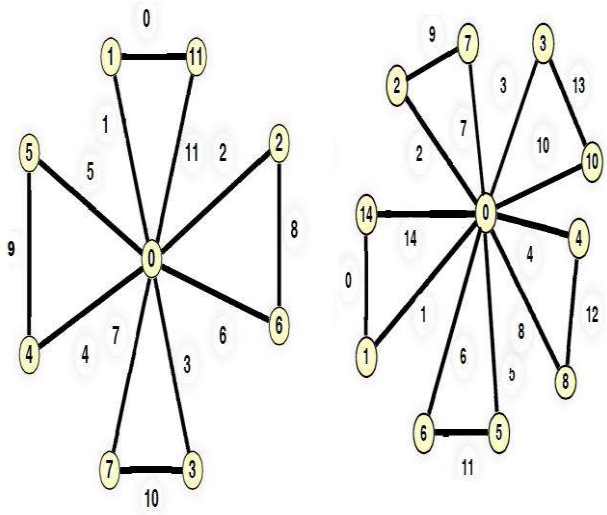
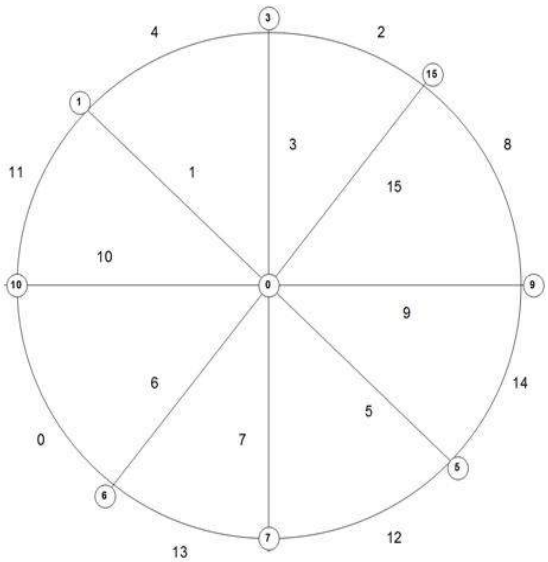


Fig 23: Harmonious Labeling of F_4 and F_5

3.11 All Wheel $W_n = C_n + K_1$ ($n \geq 3$) is harmonious.

We have shown the harmonious labeling of W_8 and W_{12}



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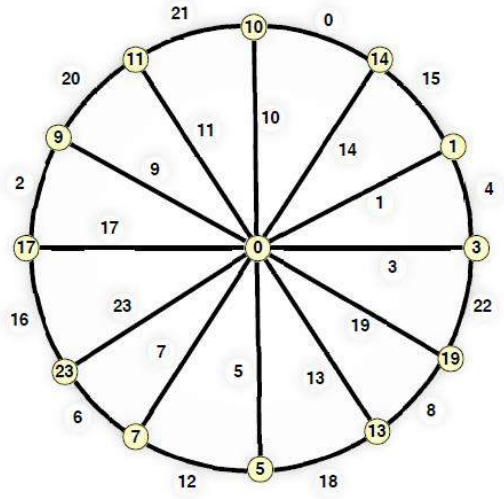

Fig 24: Harmonious Labeling of W_8 and W_{12}

4. Summary of Harmonious Graphs

We have summarized the families of familiar graphs which are harmonious.

Table 1. Summary of Harmonious Graphs

Graph	Harmonious
Trees	Harm. If n ≤ 26
Cycles C_n	Harm. Iff n is odd
Wheels W_n	Harm.
Helms	Harm.
Webs	Harm.
K_4 x P_n	Harm.
K_n	Harm. Iff n ≤ 4
$K_{m,n}$	Harm. Iff m or n = 1
Fans $f_n = P_n + K_1$	Harm.
Petersen P(n,k)	Harm.

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