Development of a Software for Analysis of Two Dimensional Nonlinear Elastic Problems Using Finite Element Method

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ABSTRACT

Present analytical theories involves determination of true stress and true strain values by the measurement of c/s area and the loads at that c/s areas. By knowing these values the stresses are determined and the true strains by knowing change in determination at each level or state. The concept of linearization and the directional derivative will be used for developing finite element software. The programming will be done in FORTRAN language.

Keywords

Finite Element, Linearization, True Strain, True Stress.

1. INTRODUCTION

The finite element method is a product of a digital computer age. The method can be symmetrically programmed to accommodate a complex and difficult problem of nonhomogeneous, nonlinear stress-strain behavior and complicated boundary conditions [1].

The finite element method is a procedure where by the continuum behavior described at infinity of points is approximated in terms of a finite number of points called nodes located at specific points in the continuum. The following steps summarize the finite element analysis procedure 1) Discretization of the continuum, 2) Selection of displacement models, 3) Derivation of element stiffness matrix using a vibrational principle , 4) Assembly of the algebraic equation for the overall discretized continuum ,5) Solution for unknown displacement ,6) Computation of the element strain and stresses from the nodal displacement [2].

1.1 Kinematics

Nonlinear kinematics quantities are linearized in preparation for inclusion in the linearized equilibrium equation that from the basic of the Newton-Raphson solution to the finite element equilibrium equations [3].

Consider a small displacement u(x) from the current configuration $x = \varphi_1(x) = \varphi(x,t)$ as shown in Fig.1

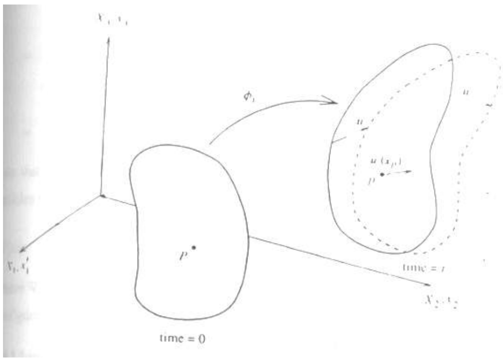

Fig.1: Linearized Kinematics

The deformation gradient F can be linearized in the direction of u at this position as

$$DF(\varphi) = \frac{d}{d\in}\bigg|_{\in=0} F(\emptyset_t + \in u)$$

$$= \frac{d}{d\in}\bigg|_{\in=0} \frac{d(\varphi_t + \in u)}{dx}$$

$$= \frac{d}{d\in}\bigg|_{\in=0} \left[\frac{d\varphi_t}{dx} + \in \frac{du}{dx}\right]$$

$$= \frac{du}{dx}$$

$$= (\nabla u)F$$

(1)

Using equation (1) and the product rule, the lagrangian strain can be linearized at the current configuration in the direction u as [4],

$$DE[u] = \frac{1}{2}(F^T DF[u] + DF[u])$$

(2)

$$= \frac{1}{2}(F^T \nabla F + F^T (\nabla u)^T F)$$

$$= \frac{1}{2} F^T |\nabla u + (\nabla u)^T| F$$

DE [u] can be interpreted as the pull back of the small strain tensor $\in$ as

$$DE[u] = \varphi^{-1}[\in] = F^T \in F$$

In particular, if the linearization of E is performed at the initial material configuration that is when x=X and therefore F=I , then

$$DE_0[u] = \in$$

Similarly the right and left Cauchy-Green deformation tensors can be linearized to give

$$DC[u] = 2F^T \in F$$

$$Db[u] = (\nabla u)b + b(\nabla u)^T$$

(3)

1.2 Linearized Volume Change

The volume change in terms of Jacobean J as

Dv=Jdv

J=det F

(4)

The directional derivative of J with respect to an increment u in the spatial configuration is,

$$DJ[u] = Ddet(F)(DF)[u]$$

Recalling the directional derivative of the determinant and the linearization of F from (1) gives,

$$DJ[u] = J\, tr\left(F^{-1}\frac{du}{dx}\right)$$

$$= J\, tr\nabla U$$

$$= J\, Div\, u$$

Finally, the directional derivative of the volume element in the direction of u emerges from equation (3) as,

$$D(dv)[u] = tr\, e\, dv$$

(5)

1.3 Discretization of the Linearized Equilibrium Equation

Equation R(x) = T(x) – F(x) = 0 represents a set of nonlinear equilibrium equation with the current nodal position as unknowns. The solution of these equations is achieved using a Newton-Raphson iterative procedure that involves the discretization of the linearized equilibrium equations. For notational convenience the virtual work equation is split into internal and external work component as,

$$\delta W(\varphi, \delta v) = \delta W_{int}(\varphi, \delta v) - \delta W_{ext}(\varphi, \delta v)$$

This can be linearized in the direction u to give

$$D\delta W(\varphi, \delta v)[u] = D\delta W_{int}(\varphi, \delta v)[u] - D\delta W_{ext}(\varphi, \delta v)[u]$$

Where the linearization of the internal visual work can be further subdivided into constitutive and initial stress component as,

$$D\delta W_{int}(\varphi, \delta v)[u] = D\delta W(\varphi, \delta v)[u] + D\delta W_{ext}(\varphi, \delta v)[u]$$

$$= \int \delta d : c : \epsilon\, dv + \int \sigma : [(\nabla u)^T \nabla \delta v] dv$$

Before continuing with the discretization of the linearized equilibrium equation, it is worth reiterating the general discussion of section to inquire in more detail why this is likely to yield a tangent stiffness matrix. Equation $\delta W^{(e)}(\varphi, Na\delta Va) = \delta Va.\left(T_a^{(e)} - F_a^{(e)}\right)$ essentially express the contribution of the nodal equivalent forces $T_a^{(e)}$ and $F_a^{(e)}$ to the overall equilibrium of node a. Change in the nodal equivalent forces $T_a^{(e)}$ and $F_a^{(e)}$ at node a , due to a change ub in the current position of node b as

$$D\delta W^{(e)}(Na, \delta Va)[Nb\, Ub] = D\left(\delta Va\left(T_a^{(e)} - F_a^{(e)}\right)\right)[Nb\, Ub]$$

$$= \delta Va.D\left(T_a^{(e)} - F_a^{(e)}\right)[Nb\, Ub]$$

$$= \delta Va.K_{ab}^{(e)} ub$$

The relation between changes in forces at node a due to changes in the current position of node b is furnished by the tangent stiffness matrix $K_{ab}^{(e)}$, which is clearly seen to derive from the linearization of the virtual work equation. In physical terms the tangent stiffness provides the Newton-Raphson procedure with the operator that adjusts current nodal position so that the deformation dependent equivalent nodal forces tend towards being in equilibrium with the external equivalent nodal forces.

1.4 Constitute Component Matrix Form

The constitute contribution to the liberalized virtual work equation for element (e) can alternatively be expressed in matrix notation by defining the small strain vector ε as,

$$\underline{\epsilon} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{13}, 2\varepsilon_{14}]^T$$

$$\varepsilon = \sum_{a=1}^{n} B_a u_a$$

The constitute component of the linearzed virtual work can now be rewritten in matrix vector notation as [5],

$$D\delta We(\pounds\varphi, \delta v)[u] = \int \delta d^T D\underline{\varepsilon}\, dv$$

Where the spatial constitute matrix D is constructed from the component of the fourth order tensor c by equating the tensor product $\delta d : c : \varepsilon$ to the matrix product of D to give after some algebra,

$$D=1/2\begin{bmatrix} 2c_{1111} & 2c_{1122} & 2c_{1133} & c_{1112}+c_{1121} & c_{1113}+c_{1131} & c_{1123}+c_{1132} \\ & 2c_{2222} & 2c_{2233} & c_{2212}+c_{2221} & c_{2213}+c_{2231} & c_{2223}+c_{2232} \\ & & 2c_{3333} & c_{3312}+c_{3321} & c_{3313}+c_{3331} & c_{3323}+c_{3332} \\ & \text{sym.} & & c_{1212}+c_{1221} & c_{1213}+c_{1231} & c_{1223}+c_{1232} \\ & & & & c_{1313}+c_{1331} & c_{1323}+c_{1332} \\ & & & & & c_{2323}+c_{2332} \end{bmatrix}$$

In a particular case of neo-Hooken material, D becomes [6]

$$D=\begin{bmatrix} \lambda'+2\mu' & \lambda' & \lambda' & 0 & 0 & 0 \\ \lambda' & \lambda'+2\mu' & \lambda' & 0 & 0 & 0 \\ \lambda' & \lambda' & \lambda'+2\mu' & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu' & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu' & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu' \end{bmatrix}$$

2. SAMPLE PROBLEM

A two dimensional example has been chosen to study many diverse features of the finite element Large Hyper elasticity problem. Same example is studied under various items explained in notes and indicated in input files [7].

The following four cases are considered.

Case 1: 2-d example with 4-noded bilinear quadrilateral elements with point and pressure loads

Case 2: 2-d example with 3-noded linear triangle elements with point and pressure loads

Case 3: 2-d example with 3-noded linear triangle elements without point and pressure loads

Case 4: 2-d example with 3-noded linear triangle elements without pressure loads with point loads only.

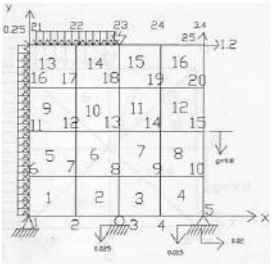

FIGURE (FOR CASE1)

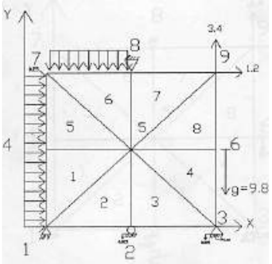

FIGURE (FOR CASE 2)

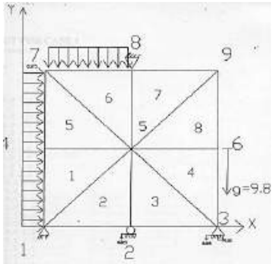

FIGURE (FOR CASE 3)

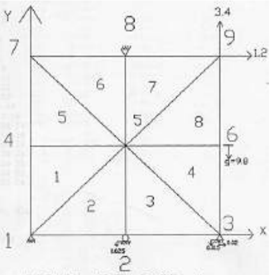

FIGURE (FOR CASE 4)

INPUT FOR CASE 3

```
2-d example
tria3
9
1 3  0.0    0.0
22   1.0   0.0
3 3  2.0    0.0
4  0   0.0
                0
50              0
1               0
7  0   0.0      0
2               0
               90 2.0   2.0

1  1  1  5  4
2     1  2  5
2     5  2  3
3  1  3  6  5
4     4  5  7
2     5  8  7
5  1  8  5  9
62    6  9  5
7  1
8  2
2     100.    100.    0.1
1  4
1.0  100.    0.1 3
2     0.0    -9.8

1.0  -0.025
0  3 -0.015
3  1 8    -0.25
2  2 4   0.25
3  2 4    -0.25
1  7 .    5.   25    1e-11
    0.0     0.0
```

OUTPUT FOR CASE 3

```
>-d  example                                   at    increment:       l,   load:
5.00
tria3
 1  3    .0000E+00    .0000E+00    -.7799E+00    .3173E+01
 2  2    .1053E+01   -.1250E+00     .0000E+00    .1729E+01
 3  3    .2100E+01   -.7500E-01    -.5140E-01    .2479E+01
 4  0    .9012E-01    .8419E+00     .1096E+01   -.1723E+01
 5  0    .1085E+01    .7847E+00     .0000E+00   -.6533E+01
 6  0    .2105E+01    .8275E+00     .0000E+00   -.1633E+01
 7  0    .1436E+00    .1754E+01     .7238E+00   -.2202E+01
 8  3    .1000E+01    .2000E+01    -.1669E+01    .6935E+01
 9  0    .1997E+01    .1790E+01     .0000E+00   -.1633E+01



   3  1     5      2     3
   4  2     3      6     5
   5  1     4      5     7
   6  2     5      8     7
   7  1     8      5     9
   8  2     6      9     5
   -  .955         2.    5018        -50.847      .10586
   3  0498        -10   .264         -23.700      .10393
   5  6798         8.   8156         -23.192      .10171
   -  .170         4.   8153         -36.496      .10869
   -  3633  -.95295                  -25.816      .10317
   -  .633         10   .706         65.058       .94181E-01
   7  1535        -22   .875         45.214       .94279E-01
   2  2915        -6.   0517         -10.015      .10141
 -  example                          at          increment :    2,    load
 0
 ria3
   9
 1    3    OOOOE+00   .OOOOE+00   -.6120E+00  .6526E+01
 2    2    1103E+01   -.2500E+00  .OOOOE+00   .4856E+01
 3    3    2200E+01   -.1500E+00  -.5877E+00  .5750E+01
 4    0    1935E+00   .7181E+00   .1941E+01   -.3646E+01
 5    0    1172E+01   .5572E+00   .OOOOE+00   -.1307E+02
 6    0    2218E+01   .6575E+00   .OOOOE+00   -.3267E+01
 7    0    3034E+00   .1552E+01   .1602E+01   -.4275E+01
 8    3    1000E+01   .2000E+01   -.3800E+01  .1150E+02
 9    0    2017E+01   .1556E+01   .OOOOE+00   -.3267E+01
```

```
  8
  1  1    1    5    4
  2  2    1    2    5
  3  1    5    2    3
  4  2    3    6    5
  5  1    4    5    7
  6  2    5    8    7
  7  1    8    5    9
  8  2    6    9    5
  -  .071      -3.  0424        -100.84     .11088
.63297         -21  .987        -50.034     .11020
  9  7610      20   .102        -56.687     .10440
  -  .374      11   .922        -74.572     .11865
  -  .491      -8.  8477        -50.795     .10624
  -  .933      6.   3829        142.78      .92421E-01
  1  .272      -37  .832        77.004      .89584E-01
  4  9344      -7.  5548        -26.696     .10415
```

3. CONCLUSION

A Fortran program is thus developed for analysis of large strain nonlinear problems. All the mathematical preliminaries required for the development of software are explained fully. This mathematics is included in the present work to larger on it is slightly difficult to grasp and complex also. This work is extension of the regular finite element programs with its own intricacies. The program can handle various element type such as 3-noded triangle, 6-noded triangle, 4-noded trilinear hexahedron elements. It is truly observed that this study gives a clear insight it to nonlinear analysis and building finite element computer software for the same. Lastly it can be said that this is an attempt to study the nonlinear continuum mechanics and the related finite element formulation and this will enhance our understanding of the commercial finite element software's for solution of such nonlinear problems. This work also has educational implications for the study of nonlinear analysis who which to start the study of such computer software. This may become guideline for them.

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