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# Absolute Continuous Function of Two Variables in $\mathbb{R}^2$ Space

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## ABSTRACT

*In this study, we provide a discussion of the absolute continuous function of two variables where one of the variables is not fixed. This research uses the definition of an absolute continuous function in* $\mathbb{R}^2$ *space with one variable that is not fixed based on Caratheodory. This research aims to examine further the properties and theorems that apply to absolute continuous functions in* $\mathbb{R}^2$ *space with one variable that is not constant, then prove these theorems. Furthermore, this paper also shows the relationship between absolute continuous functions and uniform continuous functions, continuous functions, and Lipschitz functions. In addition, it explains the algebraic properties of absolute continuous functions. This research aims to examine further the properties and theorems that apply to absolute continuous functions in* $\mathbb{R}^2$ *space with one variable that is not constant, then prove these theorems*

**Keywords:** $\mathbb{R}^2$ **space, absolute continuous function, uniform continuous functions, continuous functions**

## 1. INTRODUCTION

The absolute continuous function of two variables is a generalization of the absolute continuous function of one variable. Each concept of an absolute continuous function of two variables is derived from the concept of a continuous function of one variable. The function of two variables may be continuous on different variables in the square and not necessarily continuous on all points in the square, W. H Young and G. C Young, in their paper provide examples of continuous functions of two variables on a straight line in the square [0,1]×[0,1] [10,11]. Absolute continuous functions of two variables were previously discussed by J. Sremr, in his paper he discussed the definition of absolute continuous functions using theodory method [1,2]. Caratheodory needs to be explained here. The relationship between absolute continuous functions and finite varying functions represented by integrals by Wedie Aziz et al [3]. Finite varying functions of two variables were previously discussed by Adams and Clarkson [4,5]. A. Azocar, et all, also discuseed about the spaces of functions of two variables of bounded $\mathcal{K}\emptyset$ variation in the sense of Scrhramm – Korenblum [6]. Furthermore, J. Maly also studied absolute continuous functions for n-variables [9].

In this research, the author aims to find out (or construct) the definition of an absolute continuous function of two variables using theoretical methods and complete proof of the properties of an absolute continuous function of two variables.

## 2. LITERATURE REVIEW

Before discussing absolute continuous functions of two variables, several definitions of absolute continuous functions of one variable are discussed.

**Definition 2.1** [8] A function is said to be absolutely continuous on $[a, b]$, if for every ε>0 there exists δ(ε)>0 such that for every partition on the interval $[a, b]$, $\{[c_i, d_i] | 1 \leq i \leq n\}$, which satisfies $\sum_{i=1}^{n}(d_i - c_i) < \delta(\varepsilon)$, then

$$\sum_{i=1}^{n} |f(d_i) - f(c_i)| < \varepsilon.$$

One example of an absolute continuous function is the Lipschitz function. The Lipschitz function is a function that satisfies the Lipschitz condition as explained in definition 2.2. Every Lipschitz function is an absolute continuous function but the reverse does not apply.

**Definition 2.2** [7] Given a function $f: [a, b] \to \mathbb{R}$. The function $f$ is said to satisfy the Lipschitz condition if there is $k > 0$ such that for every $x, y \in [a, b]$ it occurs

$$|f(y) - f(x)| \leq k|y - x|.$$

Take any square $I_a^b = I \times J$ subset of the space $\mathbb{R}^2$ where $I = [x_1, x_2]$ and $J = [y_1, y_2]$ interval on $\mathbb{R}$ with $x_1 < x_2$ and $y_1 < y_2$, and $\boldsymbol{a} = (x_1, y_1), \boldsymbol{b} = (x_2, y_2)$ vector in $\mathbb{R}^2$.

Suppose $P(I)$ denotes the collection of all partitions on $I$, then $\{t_i | i = 1,2, \ldots m\} \in P(I)$, $P(J)$ denotes the collection of all partitions on $J$, then $\{s_j | j = 1,2, \ldots, n\} \in P(J)$.

If the function $f(., y): I = [x_1, x_2] \to \mathbb{R}$ defined, then it happens

$$\Delta_{10} f(t_{i+1}, s_{j+1}) = f(t_{i+1}, s_{j+1}) - f(t_i, s_{j+1}),$$

defined function $f(x, .): J = [y_1, y_2] \to \mathbb{R}$, then it happens

$$\Delta_{01} f(t_{i+1}, s_{j+1}) = f(t_{i+1}, s_{j+1}) - f(t_{i+1}, s_j),$$

and defined function $f: I \times J = [x_1, x_2] \times [y_1, y_2] \to \mathbb{R}$, then it happens

$$\Delta_{11} f(t_{i+1}, s_{j+1}) = \left(f(t_{i+1}, s_{j+1}) - f(t_{i+1}, s_j)\right) - \left(f(t_i, s_{j+1}) - f(t_i, s_j)\right),$$

Let $\mathcal{F}(D)$ represent all systems in the square $D = [a_1, a_2] \times [b_1, b_2]$ contained in the square $I_a^b$. For each square $P \in \mathcal{F}(I_a^b)$, the volume of $P$ is given by $|P|$. For example, the squares $P_1, P_2 \in \mathcal{F}(I_a^b)$ are squares that do not overlap. A square is said to not overlap if the squares do not have the same interior points. Square $P_1, P_2 \in \mathcal{F}(I_a^b)$ is said to be adjoint if squares $P_1, P_2$ not overlap each other and $P_1 \cup P_2 \in \mathcal{F}(I_a^b)$.

**Definition 2.3** [1,3] A finite function $G: \mathcal{F}(I_a^b) \to \mathbb{R}$ is said to be an additive function on a square, if for every square adjoint $P_1, P_2 \in \mathcal{F}(I_a^b)$, holds

$$G(P_1 \cup P_2) = G(P_1) + G(P_2).$$

The Lipschitz condition for a function of two variables with one variable is explained in definition 2.4.

**Definition 2.4** Given a function $f(x, .): J = [y_1, y_2] \to \mathbb{R}$. Jika $\frac{\partial f}{\partial s}$ is continuous on square $I_a^b$, then $f(x, .)$ satisfies the Lipschitz condition, if

$$|f(x,s_1) - f(x,s_2)| = \left|\frac{\partial f}{\partial s}(s_1 - s_2)\right| \le M|s_1 - s_2|.$$

For each $(x,s_1),(x,s_2) \in I_a^b$, where $M = \max_{(t,y)\in A} \frac{\partial f}{\partial s}$.

Note that $f$ satisfies the Lipschitz condition $\not\Rightarrow \frac{\partial f}{\partial s}$ is continuous.

## 3. RESULTS AND DISCUSSION

**Definition 3.1** A function $f(x,.) : J = [y_1, y_2] \to \mathbb{R}$ is said to be absolutely continuous on $J = [y_1, y_2]$ if for every ε>0 there is $\delta(\varepsilon) > 0$ such that for each partition $P(J) = \{s_j | j = 1,2,\dots,n\}$ on I which fulfills $\sum_{j=1}^{n-1}(s_{j+1} - s_j) < \delta(\varepsilon)$, holds

$$\sum_{j=1}^{n-1} |\Delta_{01} f(x, s_{j+1})| < \varepsilon.$$

**Theorem 3.2** If $f(x,.) : J = [y_1, y_2] \to \mathbb{R}$ is an absolute continuous function then $f(x,.)$ is a uniform continuous function on $J$.

**Proof.** It will be proven that the function $f(x,.)$ is uniformly continuous on $J$. Because $f(x,.)$ is absolutely continuous on $J$, then there is $P(J) = \{d_j | j = 1,2,\dots,n\}$ on $j$ which satisfies $\sum_{j=1}^{n-1}(d_{j+1} - d_j) < \delta(\varepsilon)$ holds

$$\sum_{j=1}^{n-1} |\Delta_{01} f(x, d_{j+1})| < \varepsilon.$$

Taking $m = 3$, then for every $d_1, d_2 \in J$ with $|(d_2 - d_1)| < \delta(\varepsilon)$ satisfies

$$|\Delta_{01} f(x, d_{j+1})| < \varepsilon.$$

In other words, the function $f(x,.)$ is uniform on $J$.

**Theorem 3.3** If function $f(x,.) : J \to \mathbb{R}$ absolute continuous function, then $f(x,.)$ is a continuous function.

**Proof.** Based on theorem 3.2, an absolute function is a uniform continuous function, and based on theorem (3.2), a uniform continuous function is a continuous function. So, an absolute continuous function is a continuous function.

**Theorem 3.4** If function $f(x,.) : J = [y_1, y_2] \to \mathbb{R}$ satisfies the Lipschitz condition with the Lipschitz constant $M > 0$, then function $f(x,.)$ absolute continuous on $J$.

**Proof.** It is known that $f(x,.)$ satisfies the Lipschitz condition with the constant $M > 0$, taken any way $\varepsilon > 0$ and $\delta(\varepsilon) = \frac{\varepsilon}{k}$. Let $P(J) = \{d_j | j = 1,2,\dots,n\}$ partition on $J$ satisfying $\sum_{j=1}^{n-1}(s_{j+1} - s_j) < \delta(\varepsilon)$ is obtained

$$\sum_{j=1}^{n-1} |\Delta_{01} f(x, s_{j+1})| = \sum_{j=1}^{n-1} |f(x, s_{j+1}) - f(x, s_j)| \le \sum_{i=1}^{m-1} L|t_{i+1} - t_i|$$

$$\le L \sum_{i=1}^{m-1} |t_{i+1} - t_i| \le L \cdot \frac{\varepsilon}{L} = L$$

So, funtion $f(.,y)$ is absolutely continuous on $I$.

Next, we will discuss the algebraic properties of absolute continuous functions of two variables.

**Theorem 3.5** If function $f(.,y): I = [x_1, x_2] \to \mathbb{R}$ absolute continuous function, then $|f(.,y)|$ is an absolute continuous function on $I$.

It will be proven that $|f(.,y)|$ is an absolutely continuous function on $I$. Using the triangle inequality, and because $f(.,y)$ is absolutely continuous on $I$, then

$$\sum_{i=1}^{m-1} \left||f(t_{i+1},y)| - |f(t_i,y)|\right| \leq \sum_{i=1}^{m-1} |f(t_{i+1},y) - f(t_i,y)| \leq \sum_{i=1}^{m-1} |f(t_{i+1},y) - f(t_i,y)| < \varepsilon.$$

So, the function $|f(.,y)|$ is an absolute continuous function on $I$.

**Theorem 3.6** If function $f(.,y): I \to \mathbb{R}$ absolute continuous function, then for any constant $k \neq 0$, $kf(.,y)$ is an absolute continuous function on $I$.

**Proof.** because $f(.,y): I \to \mathbb{R}$ is absolutely continuous on $I$, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that for every partition $c$ on $I = [x_1, x_2]$ that satisfies $\sum_{i=1}^{m-1}(t_{i+1} - t_i) < \delta$ such that it holds $\sum_{i=1}^{m-1} |\Delta_{10} f(t_{i+1},y)| < \frac{\varepsilon}{|k|+1} < \varepsilon$. It will be proven that $kf(.,y)$ is an absolute continuous function at $I = [x_1, x_2]$

$$\sum_{i=1}^{m-1} |\Delta_{10} kf(t_{i+1},y)| = \sum_{i=1}^{m-1} |kf(t_{i+1},y) - kf(t_i,y)| = k \sum_{i=1}^{m-1} |f(t_{i+1},y) - f(t_i,y)|$$

$$= k \sum_{i=1}^{m-1} |\Delta_{10} f(t_{i+1},y)| < \frac{\varepsilon}{|k|+1} < \varepsilon.$$

It is proven that $f(.,y)$ is an absolute continuous function on $I$.

**Theorem 3.7** If function $f(.,y)$ and $g(.,y)$ is an absolute continuous function on $I$, then $(f+g)(.,y)$ is an absolute continuous function on $I$.

**Proof.** For any $\varepsilon > 0$. Choose $\delta_f > 0$ and $\delta_g > 0$ such that by definition (3.1) holds $\sum_{j=1}^{n-1} |\Delta_{10} f(t_{i+1},y)| < \frac{\varepsilon}{2}$ and $\sum_{j=1}^{n-1} |\Delta_{10} g(v_{i+1},y)| < \frac{\varepsilon}{2}$. Defined $\delta = \min\{\delta_f, \delta_g\}$. Let $\{c_i | i = 1,2,\dots m\}$ partition on $I$, which satisfies $\sum_{j=1}^{n-1}(c_{i+1} - c_i) < \delta$. By inequality of triangles, obtained

$$\sum_{i=1}^{m-1} |\Delta_{10}(f+g)(c_{i+1},y)| = \sum_{i=1}^{m-1} |(f+g)(c_{i+1},y) - (f+g)(c_i,y)|$$

$$\leq \sum_{i=1}^{m-1} \{|f(c_{i+1},y) - f(c_i,y)|\} + \sum_{i=1}^{m-1} \{|g(c_{i+1},y) - g(c_i,y)|\}$$

$$\leq \sum_{i=1}^{m-1} |\Delta_{10} f(c_{i+1},y)| + \sum_{i=1}^{m-1} |\Delta_{10} g(c_{i+1},y)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, $(f+g)(.,y)$ is an absolute continuous function on $I = [x_1, x_2]$.

**Theorem 3.8** If the functions $f(.,y)$ and $g(.,y)$ are absolutely continuous on $I$, then $fg(.,y)$ is the absolute continuous function on $I$.

**Proof.** Because the functions $f$ and $g$ are absolute continuous functions on $I$ based on theorem (3.2) both functions are uniformly continuous, and from theorem (3.3) the function is continuous at $I$. Based on theorem (3.4), the functions $f$ and $g$ have a maximum value at $I$. Suppose $M_f+\frac{1}{2}$ is the maximum value of $f$, then $M_f+\frac{1}{2}\geq|f(t,y)|$ for each $t\in I$. And suppose $M_g+\frac{1}{2}$ is the maximum value of $g$, then $M_g+\frac{1}{2}\geq|g(t,y)|$ for each $t\in I$. Taking $\varepsilon>0$, choose $\delta_f>0$ such that $\sum_{j=1}^{n-1}|\Delta_{10}f(t_{i+1},y)|<\frac{\varepsilon}{2M_g+1}$. And choose $\delta_g>0$ such that $\sum_{j=1}^{n-1}|\Delta_{10}f(t_{i+1},y)|<\frac{\varepsilon}{2M_f+1}$. Define $\delta=\min\{\delta_f,\delta_g\}$ and take the partition $P(I)$ on $I$, which stasfies $\sum_{j=1}^{n-1}(t_{i+1}-t_i)<\delta$, then

$$\sum_{i=1}^{m-1}|\Delta_{10}(fg)(t_{i+1},y)|=\sum_{i=1}^{m-1}|(fg)(t_{i+1},y)-(fg)(t_i,y)|$$

$$\leq\left(M_g+\frac{1}{2}\right)\sum_{i=1}^{m-1}|[f(t_{i+1},y)-f(t_i,y)]|+\left(M_f+\frac{1}{2}\right)\sum_{i=1}^{m-1}|[g(t_{i+1},y)-g(t_i,y)]|$$

$$<\left(M_g+\frac{1}{2}\right)\cdot\frac{\varepsilon}{2M_g+1}+\left(M_f+\frac{1}{2}\right)\cdot\frac{\varepsilon}{2M_f+1}$$

$$<\left(M_g+\frac{1}{2}\right)\cdot\frac{\varepsilon}{2\left(M_g+\frac{1}{2}\right)}+\left(M_f+\frac{1}{2}\right)\cdot\frac{\varepsilon}{2\left(M_f+\frac{1}{2}\right)}$$

$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

So, $fg(.,y)$ is an absolute continuous function on $I$.

The discussion above is analogous to the function when the variable $x$ is running, namely $f(.,y)$.

## 4. CONCLUSION

An absolute continuous function of two variables is the development of an absolute continuous function in space R. The results obtained from this research are that every absolute continuous function of two variables is a continuous function and a Lipschitz function. Several algebraic properties also apply to absolute continuous functions, including absolute functions, multiplication with scalars, addition and multiplication of two functions.

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